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Power of Cochran's test in Behrens-Fisher problems

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George Nicholas Lauer II

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Suppose we have a random sample of size n_1, x_{1j}, $j = 1,2,3, \ldots, n_1$, from $N(\mu_1,\sigma_1^2)$ and a second random sample of size n_2, x_{2j}, $j = 1,2,3, \ldots, n_2$, from $N(\mu_2,\sigma_2^2)$. It is desired to test the hypothesis $H_0:\mu_1=\mu_2$ against $H_1:\mu_1\neq\mu_2$ when no assumptions are made regarding σ_1^2 and σ_2^2. This famous testing problem is known as the Behrens-Fisher problem. No universally accepted testing procedure now exists for this problem although certain exact procedures and simpler approximate procedures have been proposed. This thesis investigates an approximate test procedure which we term Cochran's test.

The nominal level α Cochran's test has a critical region

$$\frac{|\bar{x}_1-\bar{x}_2|}{(s_1^2/n_1+s_2^2/n_2)^{\frac{1}{2}}} > \frac{w_1t_1+w_2t_2}{w_1+w_2},$$

where $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$, $s_i^2 = \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2/(n_i-1)$, $w_i = s_i^2/n_i$,

and t_i is the $100(1-\frac{1}{2}\alpha)\%$ point of the t-distribution with n_i-1 degrees of freedom for $i=1,2$. A form of the distribution function of this test statistic is written as a multiple integral and then transformed to an expression which facilitates numerical computations for specific values of n_1 and n_2. Size and power studies for several small sample combinations are then carried out to determine the behavior of the test for various

values of $R=\sigma_1^2/\sigma_2^2$. Denote $S(R)$ as the size function for fixed R. It is found that in a practical sense, the test is uniformly conservative for $\alpha=.05$ in that $\sup_R S(R)-\alpha<.0001$ for the cases studied.

The univariate Behrens-Fisher problem is also considered when a preliminary F-test of level α_0 is used for $H_{00}:\sigma_1=\sigma_2=\sigma_0$. If H_{00} is accepted then s_1^2 and s_2^2 are pooled to estimate σ_0^2 and the standard t-test is used to make a final test of $H_{10}:\mu_1=\mu_2$. If H_{00} is rejected we use the Cochran's test for H_{10}. The distribution function of the statistic in the testing procedure is a natural extension of that for the single Cochran's test. Numerical computations are carried out for small samples and it is shown that for a proper choice of α_0 this procedure achieves a higher power than the single Cochran's test of the same size.

Finally, the multivariate extension of Cochran's test is examined. Here, our samples are made up of the pxl vectors $\underline{x}_{ij}$, i=1,2 and j=1,2,3, ... , n_i. The test of $H_0:\underline{\mu}_1=\underline{\mu}_2$ has a critical region

$$(\overline{\underline{x}}_1-\overline{\underline{x}}_2)'(S_1/n_1+S_2/n_2)^{-1}(\overline{\underline{x}}_1-\overline{\underline{x}}_2) > \left[\frac{W_1T_1+W_2T_2}{W_1+W_2}\right]^2,$$

where $\quad \overline{\underline{x}}_i = \sum_{j=1}^{n_i} \underline{x}_{ij}/n_i,\ S_i = \sum_{j=1}^{n_i} (\underline{x}_{ij}-\overline{\underline{x}}_i)(\underline{x}_{ij}-\overline{\underline{x}}_i)'/(n_i-1),$

$$W_i = |S_i/n_i|,\ T_i = \left[\frac{p(n_i-1)}{(n_i-p)}\ F_\alpha(p,n_i-p)\right]^{1/2},$$

and $F_\alpha(p,n_i-p)$ is the $100(1-\alpha)\%$ point of the F-distribution with $(\nu_1,\nu_2) = (p,n_i-p)$ degrees of freedom for i=1,2. Monte Carlo techniques are employed to make an empirical analysis of the size and power behavior for the bivariate case. Results tend to be quite consistent with the findings in the univariate case.

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I. INTRODUCTION

A. Statement of the Problem

1. Univariate population

Suppose we have a random sample of size n_1, x_{11}, x_{12}, x_{13}, ... , x_{1n_1}, from $N(\mu_1, \sigma_1^2)$ and an independent random sample of size n_2, x_{21}, x_{22}, x_{23}, ... , x_{2n_2}, from $N(\mu_2, \sigma_2^2)$. It is desired to test $H_0:\mu_1=\mu_2$ against $H_1:\mu_1\neq\mu_2$. The test statistic to be used will depend on the knowledge about σ_1 and σ_2. If σ_1 and σ_2 are both known a normal test is used. If $\sigma_1=\sigma_2$ but both are unknown a t-test is commonly used. Finally, if $\sigma_1\neq\sigma_2$ and both are unknown then we are confronted with the Behrens-Fisher problem. There is no universally accepted testing procedure for this problem although an array of tests has been developed and will be discussed in the Review of Literature. The procedure of prime interest in this study is that suggested by Cochran (see Cochran and Cox, 1957, p. 101, and Cochran, 1964). This testing method will be referred to as Cochran's test (CT) and is described by the following.

Define

$$\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i,$$

$$s_i^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2/f_i, \quad i=1,2,$$

where $f_i=n_i-1$. Also, let t_i be the $100(1-\frac{1}{2}\alpha)$ cumulative percentage point of the t-distribution with f_i degrees of freedom. The CT of $H_0:\mu_1=\mu_2$ against $H_1:\mu_1\neq\mu_2$ is to reject H_0 if

$$v = \frac{|\bar{x}_1-\bar{x}_2|}{(s_1^2/n_1+s_2^2/n_2)^{\frac{1}{2}}} > \frac{w_1t_1+w_2t_2}{w_1+w_2}, \tag{1.1}$$

where $w_i=s_i^2/n_i$, $i=1,2$. The distribution of this test statistic depends on the nuisance parameter, $R=\sigma_1^2/\sigma_2^2$, and therefore (1.1) is an approximate level α test procedure. However, it is asymptotically a normal test which is a result we would expect of a good approximate procedure. The fact that the CT statistic can be easily computed with reference only to standard t-tables has been reason for its inclusion in many statistical methods texts as a common Behrens-Fisher test. Cochran (1964) compared his procedure to that developed by Fisher (1935). Other than this, little has been written concerning the behavior of the size and power of this test. Chapter II makes a study of the CT in this connection.

The relative merits of using the CT in an incompletely specified model will also be a topic under investigation in this thesis. For this procedure a preliminary test of significance (PTS) will be made of the hypothesis $H_{00}:\sigma_1=\sigma_2$. Two possible preliminary alternative hypotheses are considered, the unilateral and bilateral cases. The hypotheses to be tested can be written as follows.

$$\left.\begin{array}{ll} H_{00}:\sigma_1=\sigma_2 & H_{01}:\sigma_1>\sigma_2 \\ H_{10}:\mu_1=\mu_2 & H_{11}:\mu_1\neq\mu_2 \end{array}\right\} \text{unilateral,}$$

$$\left.\begin{array}{ll} H_{00}:\sigma_1=\sigma_2 & H_{01}:\sigma_1\neq\sigma_2 \\ H_{10}:\mu_1=\mu_2 & H_{11}:\mu_1\neq\mu_2 \end{array}\right\} \text{bilateral.} \tag{1.2}$$

To make the bilateral test of H_{00}, we use a two-sided F-test of level α_0 with critical points d_1 and d_2. When H_{00} is accepted, i.e.,

$$d_1 < s_1^2/s_2^2 \leq d_2,$$

we use a standard t-test of level α_1 to test $H_{10}:\delta=0$, where $\delta=\mu_1-\mu_2$. H_{10} is then rejected if

$$u = \frac{|\bar{x}_1-\bar{x}_2|}{s_p(1/n_1+1/n_2)^{\frac{1}{2}}} > t, \tag{1.3}$$

where $s_p^2=(f_1s_1^2+f_2s_2^2)/(n_1+n_2-2)$, and t is the $100(1-\frac{1}{2}\alpha_1)$ % point of the t-distribution with n_1+n_2-2 degrees of freedom.

When H_{00} is rejected in the PTS, i.e.,

$$s_1^2/s_2^2 \leq d_1 \text{ or } s_1^2/s_2^2 > d_2,$$

we use the CT of nominal level α_2 to test H_{10}. Then H_{10} is rejected if (1.1) holds true. The unilateral PTS procedure is the same as above with $d_1=0$ and $d_2=d_0$ where d_0 is chosen such that the F-test is of level α_0. The levels α_0, α_1, and α_2 can be selected in various ways, each way giving rise to a procedure having certain desirable properties.

The analysis of PTS procedures is the topic of discussion in Chapter III. A computational form for the distribution of the test statistic will first be derived. Then, after some general discussion on the behavior of the test, empirical studies will be carried out for several cases and compared to other work in this area.

2. Multivariate population

The multivariate Behrens-Fisher problem is similar in form to the univariate case. Suppose we have a random sample of size n_1, containing the px1 vectors $\underline{x}_{11}, \underline{x}_{12}, \underline{x}_{13}, \cdots, \underline{x}_{1n_1}$, from $N_p(\underline{\mu}_1,\Sigma_1)$. Also available is an independent random sample of size n_2, containing the px1 vectors $\underline{x}_{21}, \underline{x}_{22}, \underline{x}_{23}, \cdots, \underline{x}_{2n_2}$, from $N_p(\underline{\mu}_2,\Sigma_2)$. It is desired to test $H_0:\underline{\mu}_1=\underline{\mu}_2$ against $H_1:\underline{\mu}_1\neq\underline{\mu}_2$. As in the univariate case, the test statistic to be used will depend on the knowledge about the population covariance matrices, Σ_1 and Σ_2. If these are both known a chi-square test is used with critical region given by

$$(\overline{\underline{x}}_1-\overline{\underline{x}}_2)'(\Sigma_1/n_1+\Sigma_2/n_2)^{-1}(\overline{\underline{x}}_1-\overline{\underline{x}}_2) > \chi^2_\alpha(p), \tag{1.4}$$

where $\overline{\underline{x}}_i = \sum_{j=1}^{n_i} \underline{x}_{ij}/n_i$, i=1,2, and $\chi^2_\alpha(p)$ is the $100(1-\alpha)$ % point of the chi-square distribution with p degrees of freedom.

In many cases $\Sigma_1=\Sigma_2=\Sigma$ but they are unknown. Then Hotelling's T^2-statistic is used to test H_0 and the critical region is given by

$$(\overline{\underline{x}}_1-\overline{\underline{x}}_2)'S_E^{-1}(\overline{\underline{x}}_1-\overline{\underline{x}}_2) > T^2_\alpha(p,n_1+n_2-p-1), \tag{1.5}$$

where we define

$$S_i = \frac{1}{f_i}\sum_{j=1}^{n_i} (\underline{x}_{ij}-\overline{\underline{x}}_i)(\underline{x}_{ij}-\overline{\underline{x}}_i)', \; i=1,2,$$

and

$$S_E = \frac{f_1S_1+f_2S_2}{f_1+f_2}\left[\frac{1}{n_1}+\frac{1}{n_2}\right].$$

S_E is an unbiased estimate of $\Sigma(1/n_1+1/n_2)$, the covariance matrix of $\overline{\underline{x}}_1-\overline{\underline{x}}_2$. In (1.5),

$$T^2_\alpha(p,n_1+n_2-p-1) = \frac{p(n_1+n_2-2)}{(n_1+n_2-p-1)} F_\alpha(p,n_1+n_2-p-1),$$

where $F_\alpha(p,n_1+n_2-p-1)$ is the $100(1-\alpha)$ % point of the F-distribution with (p,n_1+n_2-p-1) degrees of freedom. Both (1.4) and (1.5) are exact level α tests.

Now if $\Sigma_1 \neq \Sigma_2$ and both are unknown, we are confronted with a multivariate Behrens-Fisher problem. Certain methods have been devised for making the test and will be discussed in the Review of Literature. We wish to investigate the multivariate extension of Cochran's test in (1.1).

A natural extension of the univariate CT to p dimensions would be to use the generalized variance $|S_i/n_i|$ in place of s_i^2/n_i and $T_{\alpha i}^2(p,n_i-p)$ in place of t_i^2, i=1,2, in (1.1). Then the multivariate Cochran's test (MCT) of nominal level α has the critical region

$$V^2 = (\underline{\bar{x}}_1-\underline{\bar{x}}_2)'S^{-1}(\underline{\bar{x}}_1-\underline{\bar{x}}_2) > \left[\frac{W_1T_1+W_2T_2}{W_1+W_2}\right]^2, \tag{1.6}$$

where we denote

$$W_i = |S_i/n_i|,$$

$$T_i = [T_{\alpha i}^2(p,n_i-p)]^{\frac{1}{2}} = \left[\frac{p(n_i-1)}{(n_i-p)} F_\alpha(p,n_i-p)\right]^{\frac{1}{2}}, \quad i=1,2,$$

and S is a pxp matrix defined by

$$S = S_1/n_1 + S_2/n_2.$$

As in the univariate problem, this test is attractive because it is asymptotically exact. It will be shown in Chapter IV that the test approaches the exact chi-square test in (1.4). The MCT is quite usable in that the computation of the statistic is relatively simple, a feature which is lacking in some of the other Behrens-Fisher tests which have been developed. With this in mind we wish to discover what size and power characteristics can be expected when this test is used. Computations to this end are carried out in Chapter IV using Monte Carlo techniques.

B. Review of Literature

1. Univariate problem

The Behrens-Fisher problem, as described in Section A-1, is named after the two men who first wrote on the problem. Fisher (1935) used a fiducial argument in his proposed solution. Sukhatme (1938) supplemented Fisher's work by computing critical points for certain selected cases. The fiducial approach has been subject to much criticism, so consequently many alternative procedures have been developed.

Welch (1937) studied the u, v, and z-statistics to test H_0. For a nominal level α these tests have critical regions

$$u = \frac{|\bar{x}_1-\bar{x}_2|}{\left[\frac{\Sigma_1+\Sigma_2}{n_1+n_2-2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\right]^{\frac{1}{2}}} > t_{12}, \tag{1.7}$$

$$v = \frac{|\bar{x}_1-\bar{x}_2|}{\left[\frac{\Sigma_1}{n_1 f_1}+\frac{\Sigma_2}{n_2 f_2}\right]^{\frac{1}{2}}} > t_{12},$$

$$z = \frac{|\bar{x}_1-\bar{x}_2|}{\left[\frac{\Sigma_1}{n_1(n_1-3)}+\frac{\Sigma_2}{n_2(n_2-3)}\right]^{\frac{1}{2}}} > t_{12}, \quad n_1,n_2>3,$$

where

$$\Sigma_i = \sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2 = s_i^2 f_i, \; i=1,2,$$

and t_{12} is the $100(1-\frac{1}{2}\alpha)$ % point of the t-distribution with n_1+n_2-2 degrees of freedom. These statistics can be written in general as

$$\frac{Z}{[\lambda_1\chi^2(f_1)+\lambda_2\chi^2(f_2)]^{\frac{1}{2}}},$$

where $Z\sim N(0,1)$. $\chi^2(f_1)$ and $\chi^2(f_2)$ are chi-square variates with f_1 and f_2 degrees of freedom, respectively. λ_1 and λ_2 are functions of n_1, n_2, and the nuisance parameter, $R=\sigma_1^2/\sigma_2^2$. Note that the u-statistic in (1.7) is the common t-statistic with pooled estimate of variance, identical to (1.3). The v-statistic, as in (1.1), has the property that the square of the quantity in the denominator is an unbiased estimate of the variance of the numerator. It is known that when $R=1$ the u-statistic should be used to test H_0. Welch (1937) discovered that when $R\neq 1$, the size of the u-test varies substantially and the v-test gives more stable size, i.e., the size varies from α less with v than u over the range of R. The z-statistic has the property that its variance is one at $R=0$ and ∞. The z-test was found to be even less dependent on R than u and v.

Since 1937 many papers have been written furthering Welch's non-fiducial approach to the Behrens-Fisher problem. Hsu (1938) found the distribution of

$$u^2 = \frac{(\bar{x}_1-\bar{x}_2)^2}{A_1\Sigma_1+A_2\Sigma_2},$$

which is equivalent to Welch's u^2 and v^2-statistics for the appropriate choices of A_1 and A_2. Hsu then found the power function of u^2 and computed size curves for Welch's u and v-tests for various small samples.

Chand (1950) considered the probability of a Type I error when R is fixed within a range. This study differed from previous investigations in that some prior knowledge of R was assumed. Chand computed size for Welch's u and v-tests for some small samples. Both one-tailed and two-tailed tests were considered for α=.01 and .05. Some power points were computed for α=.05.

Gronow (1951) obtained approximations to the moments of the distribution of Welch's u and v-statistics using Fisher's k-statistics. Gronow arrived at a result which is more computationally convenient than that of Hsu (1938). Size and power curves were computed for $(n_1,n_2) = (10,10)$ and (15,5), for α=.01 and .05.

Rosenberg (1957) derived a modified version of Welch's z-statistic to be used for the case n_1=3, n_2=5 where the z-statistic is undefined. Rosenberg's z is given by

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\left[\frac{\Sigma_1 \pi^2}{2n_1 f_1} + \frac{\Sigma_2 \pi^2}{4n_2 f_2}\right]^{\frac{1}{2}}}.$$

Rosenberg computed the size for $(n_1,n_2) = (5,5)$ and (3,5), for α=.05 using his derived distributions of the u, v, and z-statistics. He verified Welch's conclusions that v has more size stability over the range of R than has u, but less than z.

The tests discussed so far in this section are based on some statistic and critical point, t_{12}. A testing procedure based on a variable critical point, $h(s_1^2, s_2^2, \alpha)$, was first introduced by Welch (1947). He considered both a series expansion solution and a non-series approximation in his paper, although according to Bartlett (1956) "there is a permissible criticism of Welch's solutions, namely, that the existence of an exact solution in his sense has never been rigorously established." Wilks (1940) claimed that in fact an exact solution of the form

$$\Pr\left[\frac{|\bar{x}_1-\bar{x}_2|}{(s_1^2/n_1+s_2^2/n_2)^{1/2}} \geq h(s_1^2, s_2^2, \alpha)\right] = \alpha$$

is not possible but no proof was published. On the other hand, it appears that asymptotically the test would give good results. This was brought to light by Wallace (1958).

In regard to the work of Welch (1947), Aspin (1948) proceeded "(a) to extend this expansion to some further terms, (b) to investigate the numerical behaviour of the expansion in some particular cases and (c) to consider the comparative merits of a rearranged form of the expansion." The particular cases considered were $n_1=n_2=7,13,19$ for $\alpha=.05$. Critical points were tabulated for values of $g(s_1^2, s_2^2) = \lambda_1 s_1^2/(\lambda_1 s_1^2+\lambda_2 s_2^2)$ ranging from .5 to 1, where $\lambda_i=1/n_i$. Here, size is dependent on $g(s_1^2, s_2^2)$ instead of the traditional R used in earlier work. Aspin (1949) later published a more complete table of critical values. The points included were all combinations of $n_1, n_2 = 7,9,11,16,21,\infty$, where $g(s_1^2, s_2^2) = 0(.1)1$ for $\alpha=.01$ and .05 for a one-tailed test. Trickett et al. (1956) computed tables for the same specifications only for a two-tailed test.

Ura (1955) compared the power function of the Welch (1947) test to that of the t-test for the case R=1. It was shown that the power is about the same unless n_1/n_2 is large or small, or n_1 and n_2 are both small.

The above work based on Welch's varying critical point suffers from the drawback of being somewhat difficult to use in its exact form. So, many papers have been written suggesting alternative functions, $h(s_1^2,s_2^2,\alpha)$, which yield approximate tests that are easier to compute.

McCullough et al. (1960) and McCullough (1961) considered an approximate test of H_0 with critical region

$$Y(r_1,r_2) = \frac{(\bar{x}_1-\bar{x}_2)^2}{r_1\Sigma_1+r_2\Sigma_2} > 1, \tag{1.8}$$

where r_1 and r_2 are chosen such that the size can be bounded by α over all R in a specified range. The cases where it is known $1\leq R<\infty$ (unilateral) and $0<R<\infty$ (bilateral) were both considered. By setting the size of the test equal to α at the extreme values of R in each case, r_1 and r_2 can be computed. It was found

$$r_i = t_i^2/(n_i f_i), \quad i=1,2,$$

for the bilateral case.

Although the test in (1.8) appears to have a constant critical point, it can be rewritten as a test with critical region for the bilateral case given by

$$Y(r_1,r_2) = \frac{(\bar{x}_1-\bar{x}_2)^2}{s_1^2/n_1+s_2^2/n_2} > \frac{r_1\Sigma_1+r_2\Sigma_2}{s_1^2/n_1+s_2^2/n_2} = \frac{w_1t_1^2+w_2t_2^2}{w_1+w_2}. \tag{1.9}$$

Note that this is simply a Welch (1947) test with

$$h^2(s_1^2,s_2^2,\alpha) = \frac{w_1t_1^2+w_2t_2^2}{w_1+w_2}.$$

The Welch-type test (WT) in (1.9) can also be seen to be the same as the test attributed at about the same time to Banerjee (1960) for the case of two populations.

The CT in (1.1) is seen to be of the same form as the WT with

$$h(s_1^2,s_2^2,\alpha) = \frac{w_1t_1+w_2t_2}{w_1+w_2}.$$

The similarity of the WT and CT will be explored later. As mentioned previously, the only paper studying the behavior of the CT is that of Cochran (1964), who compared his test to the numerical computations carried out by Sukhatme (1938) on the Fisher (1935) test. He found that for small or moderate n_1,n_2 the approximation gives slightly too many significant results at level α=.10. At levels α=.01 and .05 it tends to be conservative.

A thorough study of an approximate degree of freedom (APDF) test was made in an unpublished work by Yao (1962). The nominal level α APDF test has critical region

$$v > t_f,$$

where

$$\frac{1}{f} = \frac{1}{f_1}\left[\frac{s_1^2/n_1}{s_1^2/n_1+s_2^2/n_2}\right]^2 + \frac{1}{f_2}\left[\frac{s_2^2/n_2}{s_1^2/n_1+s_2^2/n_2}\right]^2 ,$$

and t_f is the $100(1-\frac{1}{2}\alpha)$ % point of the t-distribution with f degrees of freedom. Yao found that this test compares quite favorably in size behavior with the Welch (1947) test.

A comparative study of many of the tests with varying critical points was made by Mehta and Srinivasan (1970). In this paper the size and power behavior of the tests developed by Banerjee, Fisher, Welch, and others were compared for small samples.

Scheffé (1943) approached the Behrens-Fisher problem from a different standpoint than any of the previously mentioned works. His confidence interval approach has the attributes that only t-tables are required and the computations involved in finding the intervals are quite simple. Also the validity does not depend on the unknown parameter, R. However, the method is sometimes criticized because it is required that the samples be in order or be randomized, and a set of statistics

$$d_i = x_{1i} - \sum_{j=1}^{n_2} c_{ij}x_{2j}, \quad i=1,2,3, \ldots , n_1, \; n_1 \leq n_2,$$

is used rather than the original observations and the set of sufficient statistics $(\bar{x}_1,\bar{x}_2,s_1^2,s_2^2)$. The randomization procedure can lead to quite varied results. Consequently, Scheffé himself concluded his method should not be used in practice (see Scheffé, 1970).

Since a t-test is used to test H_0 when $\sigma_1=\sigma_2$ where both are unknown, and a Behrens-Fisher procedure is used when $\sigma_1\neq\sigma_2$, a logical question arises. Which test does one use if it is suspected but not known that $\sigma_1=\sigma_2$? This question suggests the use of a PTS as described in Section A-1. A PTS of the equality of σ_1 and σ_2 is made using an F-test. The final test of equality of μ_1 and μ_2 is then made using a t-test or Behrens-Fisher test, whichever is indicated by the PTS. Very little has been written on the use of a PTS in this context. Most of the studies in the use of PTS procedures have been in relation to pooling sums of squares in analysis of variance problems. Paull (1950) and Bozivich et al. (1956) concerned themselves with various problems of this nature. Kale and Bancroft (1967) and Lauer (1969) studied the effects of using a PTS in determining whether or not to pool means in testing the mean of a normal population.

Studies of PTS procedures incorporated in Behrens-Fisher problems seem to be confined to the work of McCullough (1961) and Gurland and McCullough (1962). They proposed to use a level α_0 F-test to test $H_{00}:\sigma_1=\sigma_2$. If H_{00} is accepted, a standard t-test with pooled estimate of variance is used to test $H_{10}:\mu_1=\mu_2$. If H_{00} is rejected, they suggested using some constant multiple of the t-statistic to make the Behrens-Fisher test of H_{10}. They derived the size and approximate power formulas for the PTS procedure. Using these, size and power computations were carried out for (n_1,n_2) = (3,3), (3,5), (3,7), (5,3), (5,5), and (7,3). In addition, Gurland and McCullough (1962) investigated the use of a PTS procedure using the WT studied in McCullough et al. (1960) as the Behrens-Fisher test. Some size and power comparisons were made between the two PTS procedures for the unilateral case. It was discovered that neither PTS procedure was uniformly better than the other but both showed that certain improvements in size control can be made over the respective Behrens-Fisher tests made singly.

2. Multivariate problem

The multivariate Behrens-Fisher problem has been studied in some depth with most of the solutions arising quite naturally from extensions of the univariate problem. Welch (1947) pointed out that his solution for the univariate population could be extended to testing the equality of means of several populations. Although this is still in the realm of univariate statistics it was one of the first times consideration had been given to a more extensive Behrens-Fisher testing problem. The test of equality of many means is considered again in a paper by James (1951). He carried through with an exact series solution, which in its entirety is very complicated in nature and tests using all the terms of the expansion are difficult to compute. James also considered a shortened form of the expansion. Welch (1951) derived this shortened form using an alternative approach.

The multivariate Behrens-Fisher problem, where samples are obtained from multivariate normal populations with unknown covariance matrices, was studied by Bennett (1951). His work is an extension of the Scheffé (1943) univariate solution. James (1954) developed a solution to the multivariate problem which incorporates a series analogous to that in James (1951).

The APDF test was considered in a multivariate context by Yao (1962, 1965). The test of $H_0: \underline{\mu}_1 = \underline{\mu}_2$ has critical region

$$V^2 > T^2_\alpha(p, f_T), \tag{1.10}$$

where V^2 is defined in (1.6),

$$\frac{1}{f_T} = \frac{1}{f_1}\left[\frac{\underline{y}'S^{-1}S_1S^{-1}\underline{y}}{\underline{y}'S^{-1}\underline{y}}\right]^2 + \frac{1}{f_2}\left[\frac{\underline{y}'S^{-1}S_2S^{-1}\underline{y}}{\underline{y}'S^{-1}\underline{y}}\right]^2 ,$$

and

$$\underline{y} \sim N_p(\underline{\mu}_1-\underline{\mu}_2, \ddagger_1/n_1+\ddagger_2/n_2).$$

Finding a distribution of the statistic in (1.10) would be difficult if not impossible. Yao undertook a Monte Carlo study to determine the behavior of the APDF solution. For bivariate populations, size computations were carried out for $(f_1,f_2) = (6,12)$, $(12,12)$, and $(6,18)$ for nominal levels $\alpha=.01$ and $.05$. The results were compared to the James (1954) series solution. It was discovered for the cases studied that the size is controlled better with the APDF test than with the James series h_1 solution.

C. Summary

Denote the size function of the CT as $S(R)$. It is discovered in this study that $S(0)=S(\infty)=\alpha$ and for $n_1=n_2$, $\sup_R S(R)=\alpha$. In this instance, the test is a true size α test or what we term a uniformly conservative test. For $n_1\neq n_2$ it is shown that when $|n_1-n_2|\to\infty$, $\sup_R S(R)-\alpha>0$, however, this difference is small. For all combinations of $n_1,n_2 = 3,5,7,9,13,21$ and for $\alpha = .01, .05, .10$ it is found that $\sup_R S(R)-.01<.0006$, $\sup_R S(R)-.05<.0001$, and $\sup_R S(R)-.10=0$, respectively. The CT tends to be extremely conservative for values of R near one when n_1 and n_2 are small. However, this disadvantage is overcome as n_1 and n_2 increase. When the CT is compared to the markedly similar WT, it is seen that the CT is better in both size and power characteristics.

In the preliminary testing procedures when the equality of variances is uncertain, we denote the size function by $S_X(R)$. If we use a procedure whereby $\alpha_2=\alpha$, then $S_X(0)=S_X(\infty)=\alpha$ when the CT is used as the Behrens-Fisher statistic. α_1 can be chosen such that $S_X(1)=\alpha$ and an optimal α_0, denoted by α_0^*, is selected to obtain the best size α test. This procedure reduces the extreme conservativeness which is characteristic of the CT for small samples. With a simpler procedure where $\alpha_1=\alpha_2=\alpha$, α_0^* can be chosen to give a size α test which is uniformly better than the single CT. For $n_1=n_2=n>3$, α_0^* is approximated by $\alpha_0^*=5/n^2$. This value fits the empirical results for $n = 5,7,9,13$. For $n=3$, $\alpha_0^*=.47$ is recommended.

The MCT is studied empirically for bivariate populations. The samples investigated are $(f_1,f_2) = (6,6), (6,12), (6,18)$, and $(12,12)$. It appears that the bivariate MCT behaves in about the same manner as the univariate CT. As $R \to 0,\infty$ the size of the test approaches α and for $0 < R < \infty$, the test tends to be conservative.

II. THE SIZE AND POWER OF COCHRAN'S TEST

A. Computational Formulas for the Power Function

The critical region of the CT of $H_0:\mu_1-\mu_2=\delta=0$ against $H_1:\delta\neq 0$ is, from (1.1),

$$v > \frac{w_1t_1+w_2t_2}{w_1+w_2}.$$

If R is known, the size of the test, which is dependent on R, can be written as

$$\begin{aligned} S(R) &= \Pr[\text{reject } H_0 \mid \delta=0,R] \\ &= \Pr\left[v^2 > \left[\frac{w_1t_1+w_2t_2}{w_1+w_2}\right]^2 \mid \delta=0,R\right]. \end{aligned}$$

When R is unknown, S(R) is a variable which, for the sake of discussion, will be referred to as the size function or "size", although true size is really a single value given by $\sup_R S(R)$.

It is known that

$$s_i^2f_i/\sigma_i^2\sim\chi^2(f_i), \quad i=1,2,$$

$$\frac{(\bar{x}_1-\bar{x}_2)^2}{\sigma_1^2/n_1+\sigma_2^2/n_2} \sim \chi^2(1),$$

and that all three of these chi-squares are independent. Substituting these into S(R) we obtain

$$S(R) = \Pr\left[\frac{c\chi^2(1)}{\nu_1\chi^2(f_1)+\nu_2\chi^2(f_2)} > \phi^2(\chi^2(f_1),\chi^2(f_2),\alpha)\right], \qquad (2.1)$$

where

$$c=R/n_1+1/n_2,\ \nu_1=R/(n_1f_1),\ \nu_2=1/(n_2f_2),$$

$$\phi^2(\chi^2(f_1),\chi^2(f_2),\alpha) = \left[\frac{\nu_1t_1\chi^2(f_1) + \nu_2t_2\chi^2(f_2)}{\nu_1\chi^2(f_1) + \nu_2\chi^2(f_2)}\right]^2. \qquad (2.2)$$

So given α, n_1, n_2, and R, the size of the CT is specified.

The power of the test can be written as a function of R in a manner similar to the size. Denoting power by $P(R,\lambda)$ it is seen that

$$\begin{aligned} P(R,\lambda) &= \Pr[\text{reject } H_0 \mid \delta] \\ &= \Pr\left[v^2 > \left[\frac{w_1t_1+w_2t_2}{w_1+w_2}\right]^2 \;\middle|\; \delta\right] \\ &= \Pr\left[\frac{c\chi'^2(1,\lambda)}{\nu_1\chi^2(f_1) + \nu_2\chi^2(f_2)} > \phi^2(\chi^2(f_1),\chi^2(f_2),\alpha)\right], \end{aligned}$$

where $\chi'^2(1,\lambda)$ represents a non-central chi-square variate with one degree of freedom and non-centrality parameter, λ. It is easily verified that

$$\lambda = \frac{\delta^2}{\sigma_1^2/n_1+\sigma_2^2/n_2} = \left[\frac{\delta}{\sigma_2}\right]^2\left[\frac{n_1n_2}{Rn_2+n_1}\right].$$

McCullough (1961) showed that the approximation of Patnaik (1949) gives sufficient accuracy for the power of the specific cases under consideration. This approximation will be used throughout to simplify the computations and to facilitate comparisons with McCullough's results.

Patnaik's approximation uses a central chi-square distribution to approximate the non-central chi-square distribution. Let $\chi'^2(1,\lambda)$ be approximated by $r\chi^2(s)$. Then r and s are found to be $(1+2\lambda)/(1+\lambda)$ and $(1+\lambda)^2/(1+2\lambda)$, respectively. When $\lambda=0$, we have $r=s=1$ so the approximation yields exact results for the size. Using the above, the size and power are expressed as

$$S(R) = P(R,0),$$

$$P(R,\lambda) \doteq \Pr\left[\frac{cr\chi^2(s)}{\nu_1\chi^2(f_1) + \nu_2\chi^2(f_2)} > \phi^2(\chi^2(f_1),\chi^2(f_2),\alpha)\right]. \quad (2.3)$$

From now on we will write the chi-square random variables as

$$X_1=\chi^2(f_1),\ X_2=\chi^2(f_2),\ X_3=\chi^2(s).$$

Also, let the joint density of these three independent chi-squares be denoted by

$$f(x_1,x_2,x_3) = k^* x_1^{q_1} x_2^{q_2} x_3^{q_3}\ e^{-\frac{1}{2}(x_1+x_2+x_3)}, \quad (2.4)$$

where

$$k^* = [\Gamma(\tfrac{1}{2}f_1)\Gamma(\tfrac{1}{2}f_2)\Gamma(\tfrac{1}{2}s)2^{\frac{1}{2}(f_1+f_2+s)}]^{-1},$$

$$q_i = \tfrac{1}{2}(f_i-2),\quad i=1,2,$$

$$q_3 = \tfrac{1}{2}(s-2).$$

1. The case when $n_1=n_2=n$

When the two sample sizes are equal, $t_1=t_2=t_0$ and it can be seen that

$$P(R,\lambda) \doteq \Pr\left[\frac{crX_3}{\nu_1X_1+\nu_2X_2} > t_0^2\right]. \tag{2.5}$$

This can be computed for particular cases by using the distribution function of Y, $F_Y(y)$, derived by McCullough *et al.* (1960). Y is any random variable of the form

$$Y = \frac{X_3}{\nu_1X_1+\nu_2X_2},$$

where ν_1 and ν_2 are positive constants. However, $F_Y(y)$ is difficult to compute when s is not an integer, i.e., $\lambda\neq 0$. We wish to consider the distribution function of Y in a more general form and to establish a framework to be used in Chapter III, so an alternative approach is taken in the development of computational formulas for the evaluation of (2.5). It is found that with appropriate transformations, (2.5) can be expressed in general closed form as the product of several finite summations and gamma functions, when both sample sizes are odd integers, greater than or equal to three.

Equation (2.5) can be written as

$$P(R,\lambda) \doteq \iiint_{A_1} f(x_1,x_2,x_3)\, dx_1dx_2dx_3.$$

The region of integration is

$$A_1 = \{(x_1,x_2,x_3):\ crx_3 > (\nu_1x_1+\nu_2x_2)t_0^2,\ x_i>0,\ \ i=1,2,3\}.$$

Let us make the transformation

$$y_1=\nu_1x_1+\nu_2x_2,\ y_2=x_2,\ y_3=x_3.$$

Then

$$x_1=(y_1-\nu_2y_2)/\nu_1,\ x_2=y_2, x_3=y_3,$$

and the Jacobian is

$$J = \begin{vmatrix} 1/\nu_1 & -\nu_2/\nu_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1/\nu_1.$$

The region of integration becomes

$$A_1 = \{(y_1,y_2,y_3):\ cry_3>y_1t_0^2,\ y_1>\nu_2y_2,\ y_i>0,\quad i=1,2,3\}.$$

Let

$$t' = cr/t_0^2.$$

Then

$$P(R,\lambda) \doteq \int_0^\infty \int_0^{t'y_3} \int_0^{y_1/\nu_2} k^*\nu_1^{-\frac{1}{2}f_1}\ (y_1-\nu_2y_2)^{q_1}$$

$$y_2^{q_2}y_3^{q_3} \exp\{-\tfrac{1}{2}[y_1/\nu_1 + (1-\nu_2/\nu_1)y_2+y_3]\}\ dy_2dy_1dy_3. \qquad (2.6)$$

Let $q=q_1=q_2$. When n is odd, q is an integer and we can use a binomial expansion to get

$$P(R,\lambda) \doteq \int_0^{\infty}\int_0^{t'y_3}\int_0^{y_1/\nu_2} k^*\nu_1^{-\frac{1}{2}f_1} \sum_{i=0}^{q} \binom{q}{i} y_1^i(-\nu_2)^{q-i}$$

$$y_2^{2q-i} y_3^{q_3} \exp\{-\tfrac{1}{2}[y_1/\nu_1+(1-\nu_2/\nu_1)y_2+y_3]\}\, dy_2 dy_1 dy_3. \qquad (2.7)$$

a. <u>The case $R\neq 1$</u> When $R\neq 1$, $\nu_1\neq\nu_2$ and we proceed as follows. Let

$u_1=y_1$, $u_2=c_3y_2$, $u_3=y_3$,

where $c_3=\frac{1}{2}(1-\nu_2/\nu_1)$. The Jacobian is $J=1/c_3$. Then

$$P(R,\lambda) \doteq \int_0^{\infty}\int_0^{t'u_3}\int_0^{c_3u_1/\nu_2} k^*\nu_1^{-\frac{1}{2}f_1} \sum_{i=0}^{q} \binom{q}{i} (-\nu_2)^{q-i}$$

$$(1/c_3)^{2q-i+1}\, u_1^{\,i}\, u_3^{q_3} \exp[-\tfrac{1}{2}(u_1/\nu_1+u_3)]$$

$$(u_2^{2q-i}\, e^{-u_2})\, du_2 du_1 du_3. \qquad (2.8)$$

Note that an integral of the form

$$I = \int_a^b u^n e^{-u}\, du$$

can be integrated successively to obtain

$$I = -\sum_{j=0}^{n} \frac{n!}{(n-j)!} u^{n-j} e^{-u}\Big|_a^b .$$

Using this method (2.8) becomes

$$P(R,\lambda) \doteq -\int_0^\infty \int_0^{t'u_3} \sum_{i=0}^{q} K_1(i) \left\{ \sum_{j=0}^{2q-i} \frac{(2q-i)!}{(2q-i-j)!} \right.$$

$$(c_3/\nu_2)^{2q-i-j} u_1^{2q-j} \exp[-(c_3/\nu_2+1/(2\nu_1))u_1]$$

$$\left. -(2q-i)!\, u_1^{i} \exp[-u_1/(2\nu_1)] \right\} u_3^{q_3} e^{-\frac{1}{2}u_3} du_1 du_3, \qquad (2.9)$$

where

$$K_1(i) = k^* \nu_1^{-\frac{1}{2}f_1} \binom{q_1}{i} (-\nu_2)^{q-i} (1/c_3)^{2q-i+1}.$$

(2.9) can be written as

$$P(R,\lambda) \doteq P_{11}(R,\lambda)+P_{12}(R,\lambda),$$

where

$$P_{11}(R,\lambda) = -\int_0^\infty \int_0^{t'u_3} \sum_{i=0}^{q} K_1(i) \sum_{j=0}^{2q-i} \frac{(2q-i)!}{(2q-i-j)!}$$

$$(c_3/\nu_2)^{2q-i-j} u_1^{2q-j} \exp[c_3/\nu_2+1/(2\nu_1))u_1]$$

$$u_3^{q_3} e^{-\frac{1}{2}u_3} du_1 du_3,$$

$$P_{12}(R,\lambda) = \int_0^{\infty}\int_0^{t'u_3} \sum_{i=0}^{q} K_1(i)(2q-i)!$$

$$u_1^i \exp\,[-u_1/(2\nu_1)]u_3^{q_3} e^{-\frac{1}{2}u_3}\, du_1 du_3.$$

Now make the transformations

$$z_1 = c_4 u_1, \quad z_3 = u_3,$$

$$z_1 = c_5 u_1, \quad z_3 = u_3,$$

in $P_{11}(R,\lambda)$ and $P_{12}(R,\lambda)$, respectively, where $c_4 = c_3/\nu_2 + 1/(2\nu_1)$ and $c_5 = 1/(2\nu_1)$. The Jacobians are $1/c_4$ and $1/c_5$, respectively. Then

$$P_{11}(R,\lambda) = \int_0^{\infty}\int_0^{t'c_4 z_3} \sum_{i=0}^{q} \sum_{j=0}^{2q-i} K_2(i,j)(1/c_4)^{2q-j+1}$$

$$z_3^{q_3} e^{-\frac{1}{2}z_3}(z_1^{2q-j} e^{-z_1})\, dz_1 dz_3,$$

$$P_{12}(R,\lambda) = \int_0^{\infty}\int_0^{t'c_5 z_3} \sum_{i=0}^{q} K_1(i)(2q-i)!\ (1/c_5)^{i+1}$$

$$z_3^{q_3} e^{-\frac{1}{2}z_3}(z_1^i e^{-z_1})\, dz_1 dz_3,$$

where

$$K_2(i,j) = -K_1(i)\,\frac{(2q-i)!}{(2q-i-j)!}\,(c_3/\nu_2)^{2q-i-j}.$$

Integrating these expressions as before we obtain

$$P_{11}(R,\lambda) = -\int_0^\infty \sum_{i=0}^{q} \sum_{j=0}^{2q-i} K_2(i,j)(1/c_4)^{2q-j+1}$$

$$\left\{\sum_{k=0}^{2q-j} \frac{(2q-j)!}{(2q-j-k)!}(t'c_4)^{2q-j-k}\, z_3^{2q+q_3-j-k} \exp[-(t'c_4+\tfrac{1}{2})z_3] - (2q-j)!\, z_3^{q_3} e^{-\frac{1}{2}z_3}\right\} dz_3,$$

$$P_{12}(R,\lambda) = -\int_0^\infty \sum_{i=0}^{q} K_1(i)(2q-i)!\,(1/c_5)^{i+1}$$

$$\left\{\sum_{k=0}^{i} \frac{i!}{(i-k)!}(t'c_5)^{i-k}\, z_3^{i+q_3-k} \exp[-(t'c_5+\tfrac{1}{2})z_3] - i!\, z_3^{q_3} e^{-\frac{1}{2}z_3}\right\} dz_3.$$

Let $y_3=c_6z_3$ in $P_{11}(R,\lambda)$ and $y_3=c_7z_3$ in $P_{12}(R,\lambda)$, where $c_6=t'c_4+\frac{1}{2}$ and $c_7=t'c_5+\frac{1}{2}$. The Jacobians are $1/c_6$, $1/c_7$, respectively. The final integration yields

$$P_{11}(R,\lambda) = -\sum_{i=0}^{q} \sum_{j=0}^{2q-i} K_2(i,j)(1/c_4)^{2q-j+1}$$

$$\left\{\left[\sum_{k=0}^{2q-j} \frac{(2q-j)!}{(2q-j-k)!} (t'c_4)^{2q-j-k}(1/c_6)^{2q-j-k+q_3+1}\Gamma(2q-j+q_3-k+1)\right.\right.$$

$$\left.\left.-(2q-j)!\ 2^{q_3+1}\Gamma(q_3+1)\right]\right\},$$

$$P_{12}(R,\lambda) = -\sum_{i=0}^{q} K_1(i)(2q-i)!\ (1/c_5)^{i+1}$$

$$\left\{\left[\sum_{k=0}^{i} \frac{i!}{(i-k)!} (t'c_5)^{i-k}(1/c_7)^{i-k+q_3+1}\Gamma(i+q_3-k+1)\right.\right.$$

$$\left.\left.-i!\ 2^{q_3+1}\Gamma(q_3+1)\right]\right\}. \qquad (2.10)$$

Recall that

$$P(R,\lambda) \doteq P_{11}(R,\lambda)+P_{12}(R,\lambda),$$

when $R\neq 1$, $\nu_1\neq\nu_2$, and the constants were defined in the development of (2.10) by $t'=cr/t_0^2$, $c_3=\frac{1}{2}(1-\nu_2/\nu_1)$, $c_4=c_3/\nu_2+1/(2\nu_1)$, $c_5=1/(2\nu_1)$, $c_6=t'c_4+\frac{1}{2}$, and $c_7=t'c_5+\frac{1}{2}$.

b. <u>The case R=1</u> In this instance $\nu_1=\nu_2$ so we must use another derivation since $c_3=0$ and $1/c_3$ in (2.8) is undefined. For this case, (2.7) simplifies to

$$P(1,\lambda) \doteq \int_0^{\infty}\int_0^{t'y_3}\int_0^{y_1/\nu_2} k^{*}\nu_1^{-\frac{1}{2}f_1} \sum_{i=0}^{q} \binom{q}{i}(-\nu_2)^{q-i}$$

$$y_1^i y_3^{q_3} y_2^{2q-i} \quad \exp[-\tfrac{1}{2}(y_1/\nu_1+y_3)]\ dy_2 dy_1 dy_3$$

$$= \int_0^\infty \int_0^{t'y_3} k^* \nu_1^{-\frac{1}{2}f_1} \sum_{i=0}^{q} \binom{q}{i} (-\nu_2)^{q-i} [1/(2q-i+1)]$$

$$(y_1/\nu_2)^{2q-i+1}\ y_1^i\ y_3^{q_3} \quad \exp[-\tfrac{1}{2}(y_1/\nu_1+y_3)]\ dy_1 dy_3.$$

Make the transformation

$$u_1=c_8 y_1,\ u_3=y_3,$$

where $c_8=1/(2\nu_1)$. The Jacobian is $J=1/c_8$. $P(1,\lambda)$ becomes

$$P(1,\lambda) \doteq \int_0^\infty \int_0^{t'c_8 u_3} \sum_{i=0}^{q} K_3(i) u_3^{q_3} e^{-\frac{1}{2}u_3}$$

$$u_1^{2q+1} e^{-u_1}\ du_1 du_3,$$

where

$$K_3(i)=k^* \nu_1^{-\frac{1}{2}f_1} \binom{q}{i} (-\nu_2)^{q-i} (1/c_8)^{2q+2}\ [1/(2q-i+1)](1/\nu_2)^{2q-i+1}.$$

Carrying through the integration by parts we obtain

$$P(1,\lambda) \doteq -\int_0^\infty \sum_{i=0}^{q} K_3(i) \left\{ \left[\sum_{j=0}^{2q+1} \frac{(2q+1)!}{(2q+1-j)!} \right.\right.$$

$$(t'c_8)^{2q+1-j} \; u_3^{2q+q_3+1-j} \quad \exp[-(t'c_8+\tfrac{1}{2})u_3]$$

$$\left.\left. - (2q+1)! \; u_3^{q_3} e^{-\frac{1}{2}u_3} \right] \right\} du_3.$$

Let $z_3=c_9u_3$, where $c_9=t'c_8+\frac{1}{2}$. The Jacobian is $J=1/c_9$ and integrating again, $P(1,\lambda)$ finally becomes in its entirety

$$P(1,\lambda) \doteq -k^* \nu_1^{-\frac{1}{2}f_1} (1/c_8)^{2q+2} \sum_{i=0}^{q} \binom{q}{i} (-\nu_2)^{q-i}$$

$$[1/(2q-i+1)] \; (1/\nu_2)^{2q-i+1} \left\{ \left[\sum_{j=0}^{2q+1} \frac{(2q+1)!}{(2q-j+1)!} (t'c_8)^{2q-j+1} \right.\right.$$

$$\left.\left. (1/c_9)^{2q-j+q_3+2} \; \Gamma(2q-j+q_3+2) - (2q+1)! \; 2^{q_3+1} \; \Gamma(q_3+1) \right] \right\}. \qquad (2.11)$$

So when $n_1=n_2=n\geq 3$, and n is odd, the size and approximate power of Cochran's test can be expressed by the closed form expressions in (2.10) for $R\neq 1$, and (2.11) for $R=1$. It is noted that (2.11) is a form of the approximate distribution function for the non-central t-statistic since

$$P(1,\lambda) = Pr[t'^2(2(n-1),\lambda^{\frac{1}{2}}) > t_0^2],$$

where $t'^2(2(n-1),\lambda^{\frac{1}{2}})$ represents a variate from the non-central t-distribution with 2(n-1) degrees of freedom and non-centrality parameter, $\lambda^{\frac{1}{2}}$.

2. The case when $n_1 \neq n_2$

When $n_1 \neq n_2$, $P(R,\lambda)$ is more difficult to deal with since $P(R,\lambda)$ in (2.3) cannot be simplified directly. After making many unsuccessful attempts to obtain closed form expressions we take the following alternative approach for odd values of n_1 and n_2.

From (2.3), $P(R,\lambda)$ can be written in the integral form

$$P(R,\lambda) \doteq \int\int_{A_1}\int f(x_1,x_2,x_3)\ dx_1 dx_2 dx_3,$$

where

$$A_1 = \{(x_1,x_2,x_3):\ \ crx_3(\nu_1 x_1/x_2+\nu_2) > x_2(\nu_1 t_1 x_1/x_2+\nu_2 t_2)^2,$$

$$x_i > 0,\ \ i=1,2,3\}.$$

Let

$$y_1 = \nu_1 t_1 (x_1/x_2)+\nu_2 t_2,\ y_2=x_2,\ y_3=x_3.$$

Then $x_1 = y_2(y_1-\nu_2 t_2)/(\nu_1 t_1)$, $x_2=y_2$, $x_3=y_3$, and the Jacobian is $J=y_2/(\nu_1 t_1)$. Now A_1 can be written

$$A_1 = \{(y_1,y_2,y_3):\ 0 < y_2 < g(y_1)y_3,\ y_1 > \nu_2 t_2,\ y_3 > 0\},$$

where

$$g(y_1) = cr(y_1-\nu_2 t_2+\nu_2 t_1)/(t_1 y_1^2). \tag{2.12}$$

Therefore, we have

$$P(R,\lambda) \doteq \int_{\nu_2 t_2}^{\infty} \int_0^{\infty} \int_0^{\frac{g(y_1)y_3}{[k^* y_2/(\nu_1 t_1)][y_2(y_1-\nu_2 t_2)/(\nu_1 t_1)]^{q_1}}} y_2^{q_2} y_3^{q_3} \exp\{-\tfrac{1}{2}[y_2(y_1-\nu_2 t_2+\nu_1 t_1)/(\nu_1 t_1)+y_3]\}\, dy_2 dy_3 dy_1. \quad (2.13)$$

Consider the transformation

$$u_1=y_1,\ u_2=h(y_1)y_2,\ u_3=y_3,$$

where $h(y_1) = (y_1-\nu_2 t_2+\nu_1 t_1)/(2\nu_1 t_1)$. We find that $J=1/h(u_1)$. Denote $q_{12}=q_1+q_2$. Then

$$P(R,\lambda) \doteq \int_{\nu_2 t_2}^{\infty} \int_0^{\infty} \int_0^{\frac{g(u_1)h(u_1)u_3}{[k^*/(\nu_1 t_1)][1/h(u_1)]^{q_{12}+2}}} [(u_1-\nu_2 t_2)/(\nu_1 t_1)]^{q_1}\, u_3^{q_3} e^{-\frac{1}{2}u_3} (u_2^{q_{12}+1} e^{-u_2})\, du_2 du_3 du_1$$

$$= \int_{\nu_2 t_2}^{\infty} \int_0^{\infty} -[k^*/(\nu_1 t_1)][1/h(u_1)]^{q_{12}+2} [(u_1-\nu_2 t_2)/(\nu_1 t_1)]^{q_1}$$

$$\left\{\left[\sum_{i=0}^{q_{12}+1} \frac{(q_{12}+1)!}{(q_{12}+1-i)!} [g(u_1)h(u_1)]^{q_{12}+1-i} u_3^{q_{12}+1-i+q_3}\right.\right.$$

$$\left.\left.\exp[-(g(u_1)h(u_1)+\tfrac{1}{2})u_3] - (q_{12}+1)!\, u_3^{q_3} e^{-\frac{1}{2}u_3}\right]\right\} du_3 du_1. \qquad (2.14)$$

Further, make the following transformation

$$z_1 = u_1, \quad z_3 = p(u_1)u_3,$$

where $p(u_1) = g(u_1)h(u_1)+\frac{1}{2}$. Then $J = 1/p(z_1)$. Substituting this in (2.14), multiplying by J, and integrating in z_3 to a gamma function, it is found that

$$P(R,\lambda) \doteq \int_{\nu_2 t_2}^{\infty} -[k^*/(\nu_1 t_1)][1/h(z_1)]^{q_{12}+2} [(z_1-\nu_2 t_2)/(\nu_1 t_1)]^{q_1}$$

$$\left\{\left[\sum_{i=0}^{q_{12}+1} \frac{(q_{12}+1)!}{(q_{12}-i+1)!} [g(z_1)h(z_1)]^{q_{12}-i+1}\right.\right.$$

$$[1/p(z_1)]^{q_{12}-i+q_3+2}\, \Gamma(q_{12}-i+q_3+2)$$

$$\left.\left. - (q_{12}+1)!\, 2^{q_3+1}\, \Gamma(q_3+1)\right]\right\} dz_1. \qquad (2.15)$$

It was found that when functions of z_1 were combined, the result was not directly integrable to any known closed form expression. Consequently, the computational form for the power function of Cochran's test will be left in the single integral form, (2.15). Evaluation will be carried out numerically using a computer for various odd values of n_1 and n_2, when

$n_1 \neq n_2$. Valid results are also obtainable with (2.15) for $n_1 = n_2$, but the expressions in (2.10) and (2.11) are easier to use in this case.

B. The Behavior of the Size and Power

1. Theoretical development

A very necessary property of an approximate test is that as n_1 and n_2 become large, it should approach a normal test. Let $Z_{\frac{1}{2}\alpha}$ represent the $100(1-\frac{1}{2}\alpha)$ % point of the standard normal distribution. Since $n_1, n_2 \to \infty$ implies $t_1, t_2 \to Z_{\frac{1}{2}\alpha}$, $s_1^2 \to \sigma_1^2$, and $s_2^2 \to \sigma_2^2$, the CT is asymptotically a normal test. Therefore, for a CT of nominal level α

$$\lim_{n_1, n_2 \to \infty} S(R) = \alpha \qquad \forall R,$$

$$\lim_{n_1, n_2 \to \infty} P(R,\lambda) = 1 \qquad \forall R, \forall \lambda \neq 0,$$

where $S(R)$ is given by (2.1) and $S(R) = P(R,0)$.

The behavior of $S(R)$ as $R \to 0,\infty$ is apparent by examining (2.1) and (2.2). Since

$$\lim_{R \to 0} \nu_1/c = 0, \quad \lim_{R \to 0} \nu_2/c = 1/f_2,$$

$$\lim_{R \to 0} \phi^2(X_1, X_2, \alpha) = t_2^2,$$

we have

$$\lim_{R \to 0} S(R) = \Pr\left[\frac{\chi^2(1)}{X_2/f_2} > t_2^2\right]$$

$$= \Pr[t^2(f_2) > t_2^2] = \alpha,$$

where $t(f_2)$ represents a variate from the t-distribution with f_2 degrees of freedom. Similarly,

$$\lim_{R \to \infty} \nu_1/c=1/f_1, \lim_{R \to \infty} \nu_2/c=0,$$

$$\lim_{R \to \infty} \phi^2(X_1,X_2,\alpha)=t_1^2.$$

Hence,

$$\lim_{R \to \infty} S(R)=\Pr\left[\frac{\chi^2(1)}{X_1/f_1} > t_1^2\right]$$

$$= \Pr[t^2(f_1) > t_1^2] = \alpha.$$

So at the extreme values for R, the size of the CT approaches α. Analogously, as $R \to 0,\infty$, the power function, $P(R,\lambda)$, will approach the power of the t-tests with f_2 and f_1 degrees of freedom, respectively.

Some insight into the size behavior of the CT for $0<R<\infty$ can be gained by comparing it to the WT procedure, (1.9), of McCullough et al. (1960). For $n_1=n_2$, the tests are identical. Wald (1955) showed for this case that $S(0)=\alpha$, $S(\infty)=\alpha$, $S(R)$ is decreasing when $0<R<1$, $S(R)$ is increasing when $1<R<\infty$, and $S(R)$ is minimum at $R=1$. Consequently, for $n_1=n_2$ the WT and CT are uniformly conservative in the sense that the size function is bounded above by α. Hence the tests are true size α tests. Extending Wald's work for $n_1 \neq n_2$ it can be shown that the size of the WT is still bounded above by α and is decreasing for

$$R<(n_1f_1t_2^2)/(n_2f_2t_1^2)$$

when $n_1 \leq n_2$. The details of this development will not be presented here. It is easily shown that

$$\left[\frac{w_1t_1+w_2t_2}{w_1+w_2}\right]^2 \le \left[\frac{w_1t_1^2+w_2t_2^2}{w_1+w_2}\right],$$

where equality holds when $n_1=n_2$. Hence, the critical point for the CT is less than or equal to that for the WT. Thus, we would expect the size and power of the CT to be at least as high as for the WT. Due to the similarity of the CT and WT tests we would expect similar behavior over the range of R. The point in question now is whether or not the CT is uniformly conservative when $n_1 \ne n_2$. Since we have already shown $S(R) \to \alpha$ at the extremes in R, we will attempt to find out whether it approaches α from above or below.

S(R) in (2.1) can be written as

$$S(R) = \Pr[\chi^2(1) > \psi_1(X_1,X_2,R)],$$

where

$$\psi_1(X_1,X_2,R) = \frac{n_1n_2(t_1\nu_1X_1+t_2\nu_2X_2)^2}{(Rn_2+n_1)(\nu_1X_1+\nu_2X_2)}, \tag{2.16}$$

and ν_1,ν_2 are given in (2.2). Now by interchanging n_1 and n_2, t_1 and t_2, and replacing R by 1/R in (2.16), the size remains constant by the symmetry of the test. Therefore,

$$S(R) = \Pr[\chi^2(1) > \psi_2(X_1,X_2,R)],$$

where

$$\psi_2(X_1,X_2,R) = \frac{n_1n_2(t_2\nu_2X_1+t_1\nu_1X_2)^2}{(Rn_2+n_1)(\nu_2X_1+\nu_1X_2)}.$$

Then $$S(R) = \tfrac{1}{2}\Pr[\chi^2(1) > \psi_1(X_1,X_2,R)] + \tfrac{1}{2}\Pr[\chi^2(1) > \psi_2(X_1,X_2,R)]$$

$$= \tfrac{1}{2}\int_0^\infty\int_0^\infty \left\{ \int_{\psi_1(x_1,x_2,R)}^\infty [(2t)^{\frac{1}{2}}\Gamma(\tfrac{1}{2})]^{-1}e^{-\frac{1}{2}t}dt + \int_{\psi_2(x_1,x_2,R)}^\infty [(2t)^{\frac{1}{2}}\Gamma(\tfrac{1}{2})]^{-1}e^{-\frac{1}{2}t}dt \right\}$$

$f(x_1,x_2)\ dx_1dx_2,$

where $f(x_1,x_2)$ represents the joint density function of two independent chi-square variates with f_1 and f_2 degrees of freedom. By Leibnitz's rule for multiple integrals

$$\frac{d}{dR} S(R) = \tfrac{1}{2} \int_0^\infty \int_0^\infty \{-[(2\psi_1(x_1,x_2,R))^{\frac{1}{2}}\Gamma(\tfrac{1}{2})]^{-1} e^{-\frac{1}{2}\psi_1(x_1,x_2,R)}$$

$$\frac{d}{dR}\psi_1(x_1,x_2,R) - [(2\psi_2(x_1,x_2,R))^{\frac{1}{2}}\Gamma(\tfrac{1}{2})]^{-1}e^{-\frac{1}{2}\psi_2(x_1,x_2,R)}$$

$$\frac{d}{dR}\psi_2(x_1,x_2,R)\}\ f(x_1,x_2)\ dx_1dx_2. \tag{2.17}$$

If this integral is negative, we know the size function is decreasing. If the integral is positive, size is increasing. Therefore, we will find the range of R for which the sign of the integrand is the same for all x_1 and x_2. Let I be the quantity in braces in (2.17). Since we are interested only in the sign of I and are going to compare it to zero, we can multiply I by the positive quantity

$$(Rn_2+n_1)^{3/2}(2n_1n_2)^{1/2}\Gamma(1/2).$$

We thus obtain I_0 which is found to have the form

$$I_0 = -Q_1e^{-M_1}-Q_2e^{-M_2},$$

where

$$M_1 = \tfrac{1}{2}\psi_1(x_1,x_2,R),\ M_2 = \tfrac{1}{2}\psi_2(x_1,x_2,R),$$

$$Q_1 = \{[t_1n_2Rx_1^2/f_1^2-t_2n_1x_2^2/f_2^2]$$

$$+ [n_2R(t_1-2t_2)+n_1(2t_1-t_2)]x_1x_2/(f_1f_2)\}/(\nu_1x_1+\nu_2x_2)^{3/2},$$

$$Q_2 = \{[t_1n_2Rx_2^2/f_1^2-t_2n_1x_1^2/f_2^2]$$

$$+[n_2R(t_1-2t_2)+n_1(2t_1-t_2)]x_1x_2/(f_1f_2)\}/(\nu_2x_1+\nu_1x_2)^{3/2}.$$

Define $c^*=f_2(2t_1-t_2)/(f_1t_2)$ and let $R \to 0$.

a. <u>The case $n_1<n_2$</u> In this instance $t_1>t_2$ and the following hold.

$$Q_1>0,\ Q_2>0 \qquad \text{if } 1/c^*<x_1/x_2<c^*,$$

$$Q_1>0,\ Q_2\le 0,\ e^{-M_1}>e^{-M_2} \quad \text{if} \quad x_1/x_2\ge c^*,$$

$$Q_1\le 0,\ Q_2>0,\ e^{-M_1}<e^{-M_2} \quad \text{if} \quad x_1/x_2\le 1/c^*.$$

We now show $I_0<0$ in each of these regions. When $Q_1>0$, $Q_2>0$, then obviously $I_0<0$. Now we will show by contradiction that $Q_1>|Q_2|=-Q_2$ when $x_1/x_2\ge c^*$. Assume the opposite. Then $Q_1\le -Q_2$ implies

$$[-t_2n_1x_2^2/f_2^2+n_1(2t_1-t_2)x_1x_2/(f_1f_2)]/x_2^{3/2}$$

$$\le [t_2n_1x_1^2/f_2^2-n_1(2t_1-t_2)x_1x_2/(f_1f_2)]/x_1^{3/2}$$

which reduces to

$$((2t_1-t_2)/t_2)[(x_1/x_2)^{3/2}+1] \le [x_1/x_2+(x_1/x_2)^{1/2}]f_1/f_2.$$

Since $t_1>t_2$ and $f_2>f_1$ we can write

$$[(x_1/x_2)^{3/2}+1] < [x_1/x_2+(x_1/x_2)^{1/2}].$$

Let $x_1/x_2=1+\varepsilon$ where $\varepsilon>0$. Then we have

$$(1+\varepsilon)^{3/2}+1 < 1+\varepsilon+(1+\varepsilon)^{1/2}$$

$$1+\varepsilon < \varepsilon/(1+\varepsilon)^{1/2}+1$$

$$(1+\varepsilon)^{1/2} < 1$$

$$1+\varepsilon < 1.$$

This is a contradiction. So $Q_1>|Q_2|$ for all $x_1/x_2>1$, hence for $x_1/x_2\geq c^*>1$. Therefore, in this region $I_0<0$. By symmetry it can be shown $|Q_1|<Q_2$ when $x_1/x_2\leq 1/c^*$. Therefore, we conclude $I_0<0$ for all x_1/x_2, and $S(R)$ is decreasing from α at $R=0$.

b. <u>The case $n_1>n_2$</u> Here, $t_1<t_2$ and it is evident that $c^*<1$. It is easily verified that

$$Q_1<0,\ Q_2<0 \quad \forall x_1/x_2, \quad \text{if } 2t_1-t_2\leq 0,$$

$$Q_1<0 \quad \text{if } x_1/x_2<1/c^*, \quad 2t_1-t_2>0,$$

$$Q_2<0 \quad \text{if } x_1/x_2>c^*, \quad 2t_1-t_2>0.$$

We see that

$$\lim_{n_1/n_2\to\infty} c^*=0, \qquad \lim_{n_1/n_2\to\infty} 1/c^*=\infty,$$

and therefore $Q_1,Q_2<0$ for all x_1/x_2 as $n_1/n_2\to\infty$. As a result, $I_0>0$ and $S(R)$ is increasing from α at $R=0$.

In summary, in this section we have discovered that the CT is identical to the WT for $n_1=n_2$ and thereby is uniformly conservative with its minimum size at R=1. By the symmetry of the foregoing results, we have also found in general that

$$n_1<n_2 \begin{cases} \lim\limits_{R\to 0} S(R) = \alpha^- \\ \lim\limits_{n_1/n_2\to 0} \lim\limits_{R\to\infty} S(R) = \alpha^+ \end{cases}$$

$$n_1>n_2 \begin{cases} \lim\limits_{n_1/n_2\to\infty} \lim\limits_{R\to 0} S(R) = \alpha^+ \\ \lim\limits_{R\to\infty} S(R) = \alpha^- . \end{cases}$$

In the most general sense, the CT is not uniformly conservative as is the WT. However, for the vast majority of values of n_1/n_2, it may very well be conservative, although no rigorous proof of this seems feasible. From the similarity of the CT and the WT and the computation of S(R) for several specific values of n_1 and n_2, we can say that the difference $\sup\limits_{R} S(R)-\alpha$ is always small, perhaps negligible for practical work. More will be said about this in what follows.

2. Results of computations

In order to learn more about the behavior of the size and power function of Cochran's test, computations for several cases were carried out using a computer. The results are tabulated in the Appendix. Tables demonstrating the behavior of other tests are included for comparison. Extensive computations for nominal level $\alpha=.05$ are tabulated since this is one of the most commonly used levels in actual practice. Some results are also included for α=.01 and .10. Only the values $R \geq 1$ need be considered because of symmetry.

Table 1 contains the size of the standard t-test, (1.3), as R varies. Denote the test by t_R^2 to indicate the dependency on R. At R=1, size equals .05 because the test is exact in this instance. Note the large deviations above .05 when $n_1 \leq n_2$ and the wide fluctuations below .05 when $n_1 > n_2$. Furthermore, it is not immediately obvious what the upper bound on the size is. These results were a common finding in many papers on the subject and were what prompted researchers to find better tests to be used in Behrens-Fisher problems.

Tables 2 and 3 contain size computations for the WT and CT for α=.05. These results tend to support the findings of the previous section. The gradual approach to an exact level .05 test is evident as $n_1, n_2 \to \infty$. The tests are also exact as $R \to 0, \infty$ as pointed out before. The WT and CT are seen to be the same for $n_1 = n_2$. The CT exceeds the WT in size for $n_1 \neq n_2$; however, the difference is not too great when $|n_1 - n_2|$ is small. From the tables, both tests look uniformly conservative but consistent with the development of the previous section, it was found that the size function of the CT ventures above .05 slightly for large R. The tendency of size to exceed .05 appears to increase as $|n_1 - n_2|$ increases. However, even for $n_1=3$, $n_2=21$ it is found that $\sup_R S(R) - .05 < .0001$, so for all practical purposes we can consider the CT to be uniformly conservative at level α=.05.

Tables 4, 5, 6, and 7 illustrate the size behavior of the WT and CT for α=.01 and .10. Since $\sup_R S(R) - .01 < .0006$ and $\sup_R S(R) - .10 = 0$ for all the cases considered, it appears that as α increases, $\sup_R S(R)-\alpha$ becomes less significant.

Tables 8, 9, and 10 contain a power comparison between the CT, WT, and u_R^2. u_R^2 is the t-test of (1.7) where it is assumed R is known and size equals .05 for that value of R. It is an optimum test in that sense. For R=1, $t_R^2=u_R^2$ so the values in Table 8 are values for the power of the t-test. Tables 9 and 10 contain the power of the tests for R=4 and 10, respectively. Note that the power approaches one as $n_1,n_2,\lambda \to \infty$. It is interesting to note that the size and power behavior differ somewhat. For instance, for all λ considered and for the sequence R=1,4,10, we have

n_1	n_2	S(R)	P(R,λ)
3	21	increasing	decreasing
21	3	decreasing	increasing, decreasing
21	21	increasing	decreasing, increasing.

Here, the size and power behave in essentially the opposite manner. This tells us that for fixed n_1,n_2 and λ, the size and power functions do not achieve their maximums at the same value of R. Of course, if $\lambda \to 0$ the power behavior will approach that of size.

As we would expect, the power behavior of the WT and CT is the same. The power for the CT always is at least as large as that for the WT. It appears, then, that the CT is uniformly better than the WT. Both are essentially conservative but the CT has better size control and a higher power, especially when $|n_1-n_2|$ is large. The chief disadvantage of the CT, which is also inherent in the WT, is the extreme conservativeness of the test for certain values of R when n_1 and n_2 are small. What one must decide before choosing a Behrens-Fisher testing procedure is how important the bounding of Type I error really is. In certain experiments it may be critical that size does not exceed α by much. In other words, we want to be reasonably sure the test is a true size α test. In such cases, the CT would be adequate. In some experiments the bounding of Type I error may not be so essential, in which case, some of the test procedures discussed in Chapter I-B-1 may be more suitable both in size and power characteristics.

C. A Variation of Cochran's Test for the Unilateral Case

Since no assumptions were made about R for the preceding analysis, the CT as discussed was bilateral in nature. If it were desired to concern ourselves with a situation where it is known $R \geq 1$ the CT of (1.1) could be converted to a test with critical region

$$\bar{v} > \frac{w_1 t_1 + w_2 t_2'}{w_1 + w_2},$$

where t_2' is chosen to give the test size α at $R=1$. The comparable WT statistic can be found by setting $r_2 = r_2'$ such that the size is α at $R=1$. Summary results are presented in Tables 11 and 12 for the size of the two unilateral tests. For $n_1 > 3$ it appears that the unilateral CT has better size control than the WT but not by a great deal.

III. PRELIMINARY TESTING PROCEDURES INVOLVING COCHRAN'S TEST

A brief introduction to the PTS procedure was given in Chapter I-A-1. As before it is desired to test $\mu_1=\mu_2$ when two independent random samples are available, one from each of the populations $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$. It is suspected but not known with certainty that $\sigma_1=\sigma_2$ so we make a preliminary test of this hypothesis prior to testing $\mu_1=\mu_2$. The test statistic used in the final test depends on the result of the PTS. The two sets of hypotheses under consideration are given in (1.2) as

$$\left.\begin{array}{ll} H_{00}:\sigma_1=\sigma_2 & H_{01}:\sigma_1>\sigma_2 \\ H_{10}:\mu_1=\mu_2 & H_{11}:\mu_1\neq\mu_2 \end{array}\right\} \text{ unilateral,}$$

$$\left.\begin{array}{ll} H_{00}:\sigma_1=\sigma_2 & H_{01}:\sigma_1\neq\sigma_2 \\ H_{10}:\mu_1=\mu_2 & H_{11}:\mu_1\neq\mu_2 \end{array}\right\} \text{ bilateral.}$$

In the unilateral testing situation it is assumed that $R\geq 1$, while in the bilateral case $0<R<\infty$. The overall bilateral testing procedure as described in Chapter I-A-1 is to reject H_{10} if

$$u>t \qquad \text{when} \qquad d_1<s_1^2/s_2^2\leq d_2$$

or

$$v>(w_1t_1+w_2t_2)/(w_1+w_2) \text{ when } s_1^2/s_2^2\leq d_1 \text{ or } s_1^2/s_2^2>d_2.$$

The preliminary F-test is of level α_0, the t-test is of level α_1, and the CT is a nominal level α_2 test. The selection of these levels gives rise to different procedures which will be discussed in Section C.

A. The Components of Power

The following analysis is a generalization of the development in Chapter II-A. The size and power for the PTS procedure can be written in terms of two mutually exclusive components. Denote size by $S_X(R)$. Then

$$S_X(R) = \Pr[\text{reject } H_{10} \mid \delta=0] = S_1(R)+S_2(R),$$

where

$$S_1(R) = \Pr[\text{accept } H_{00} \text{ and reject } H_{10} \mid \delta=0]$$

$$= \Pr[d_1<s_1^2/s_2^2\leq d_2,\ u>t],$$

$$S_2(R) = \Pr[\text{reject } H_{00} \text{ and reject } H_{10} \mid \delta=0]$$

$$= \Pr[s_1^2/s_2^2\leq d_1 \text{ or } s_1^2/s_2^2>d_2,$$

$$v>(w_1t_1+w_2t_2)/(w_1+w_2)].$$

Formulating these expressions in terms of chi-square variates it is found that

$$S_1(R) = \Pr[a_1/R<X_1/X_2\leq a_2/R,\ \frac{\chi^2(1)}{\lambda_1X_1+\lambda_2X_2} > t^2],$$

$$S_2(R) = \Pr[X_1/X_2\leq a_1/R \text{ or } X_1/X_2>a_2/R,$$

$$\frac{c\chi^2(1)}{\nu_1X_1+\nu_2X_2} > \phi^2(X_1,X_2,\alpha_2)],$$

where

$$a_i = (f_1/f_2)d_i \quad i=1,2,$$

$$\lambda_1 = \frac{R(n_1+n_2)}{(f_1+f_2)(Rn_2+n_1)}, \quad \lambda_2 = \frac{(n_1+n_2)}{(f_1+f_2)(Rn_2+n_1)} = \lambda_1/R$$

and c, ν_1, ν_2, and $\phi(X_1,X_2,\alpha_2)$ are given in (2.2). The size of the procedure is specified once α_0, α_1, α_2, n_1, n_2, and R are given.

The power of the test can be written in essentially the same manner as the size and we have

$$P_1(R,\lambda) = \Pr[a_1/R<X_1/X_2\le a_2/R, \frac{\chi'^2(1,\lambda)}{\lambda_1X_1+\lambda_2X_2} > t^2], \tag{3.1}$$

$$P_2(R,\lambda) = \Pr[X_1/X_2\le a_1/R \text{ or } X_1/X_2>a_2/R, \frac{c\chi'^2(1,\lambda)}{\nu_1X_1+\nu_2X_2} > \phi^2(X_1,X_2,\alpha_2)]. \tag{3.2}$$

Using the method of Patnaik (1949) as before we have

$$P_1(R,\lambda) \doteq \Pr[a_1'<X_1/X_2\le a_2', \frac{rX_3}{\lambda_1X_1+\lambda_2X_2} > t^2], \tag{3.3}$$

$$P_2(R,\lambda) \doteq \Pr[X_1/X_2\le a_1' \text{ or } X_1/X_2>a_2', \frac{crX_3}{\nu_1X_1+\nu_2X_2} > \phi^2(X_1,X_2,\alpha_2)] \tag{3.4}$$

where $a_1'=a_1/R$ and $a_2'=a_2/R$.

Denote the exact power of the PTS procedure by $P_X(R,\lambda)$. We have

$$S_X(R) = P_X(R,0),$$

$$P_X(R,\lambda) = P_1(R,\lambda) + P_2(R,\lambda).$$

The size and power for the unilateral case are of the same form with $a_1=0$ and $a_2=a_0$ where a_0 is chosen such that the now one-tailed PTS is still of level α_0. Let $a_1=a_2=0$. Then the power function for the general PTS becomes

$$P_X(R,\lambda) = P_2(R,\lambda) \doteq \Pr\left[\frac{crX_3}{\nu_1X_1+\nu_2X_2} > \phi^2(X_1,X_2,\alpha_2)\right].$$

This is simply the power function for the CT, identical to (2.3) for $\alpha_2=\alpha$ and what follows is a generalization of the derivation subsequent to (2.3).

B. Formulas for the Computation of the Power

1. The first component of power

This section will derive a computational formula for $P_1(R,\lambda)$ using the same approach as in Chapter II-A. Let n_1 and n_2 be odd. The integral expression for (3.3) is

$$P_1(R,\lambda) \doteq \int\int_{A_2}\int f(x_1,x_2,x_3)\, dx_1dx_2dx_3, \tag{3.5}$$

where $f(x_1,x_2,x_3)$ is the joint density of three independent chi-squares given in (2.4). The region of integration is

$$A_2 = \{(x_1,x_2,x_3):\ a_1'x_2<x_1\leq a_2'x_2,\ rx_3>t^2(\lambda_1x_1+\lambda_2x_2),\ x_i>0\ i=1,2,3\}.$$

Let us make the transformation.

$$y_1=\lambda_1x_1+\lambda_2x_2,\ y_2=x_2,\ y_3=x_3.$$

Then $x_1=(y_1-\lambda_2 y_2)/\lambda_1$, $x_2=y_2$, $x_3=y_3$ and the Jacobian is $1/\lambda_1$. The region of integration becomes

$$A_2 = \{(y_1,y_2,y_3):\ y_1/c_2 \leq y_2 < y_1/c_1,\ 0<y_1<t^* y_3,\ y_3>0\},$$

where $c_i=a_i'\lambda_1+\lambda_2$, $i=1,2$, and $t^*=r/t^2$. Therefore (3.5) becomes

$$P_1(R,\lambda) \doteq \int_0^\infty \int_0^{t^* y_3} \int_{y_1/c_2}^{y_1/c_1} k^*\lambda_1^{-\frac{1}{2}f_1} (y_1-\lambda_2 y_2)^{q_1}$$

$$y_2^{q_2} y_3^{q_3} \exp\{-\tfrac{1}{2}[y_1/\lambda_1+(1-\lambda_2/\lambda_1)y_2+y_3]\}\, dy_2\, dy_1\, dy_3. \qquad (3.6)$$

Note that this expression is similar to that in (2.6). Using the same sequence of transformations as in the development of (2.7), (2.8), (2.9), and (2.10), for $R\neq 1$ we arrive at the result

$$P_1(R,\lambda) \doteq -k^*\lambda_1^{-\frac{1}{2}f_1} \sum_{i=0}^{q_1} \binom{q_1}{i} (-\lambda_2)^{q_1-i} \sum_{j=0}^{q_{12}-i} (1/c_3)^{j+1} \frac{(q_{12}-i)!}{(q_{12}-i-j)!}$$

$$\Big\{ \Big[(1/c_1)^{q_{12}-i-j} (1/c_4)^{q_{12}-j+1} \Big[\sum_{k=0}^{q_{12}-j} \frac{(q_{12}-j)!}{(q_{12}-j-k)!} (t^* c_4)^{q_{12}-j-k}$$

$$(1/c_6)^{q_{12}-j-k+q_3+1} \Gamma(q_{12}-j-k+q_3+1) - (q_{12}-j)!\, 2^{q_3+1}\, \Gamma(q_3+1) \Big]$$

$$- (1/c_2)^{q_{12}-i-j} (1/c_5)^{q_{12}-j+1} \Big[\sum_{k=0}^{q_{12}-j} \frac{(q_{12}-j)!}{(q_{12}-j-k)!} (t^* c_5)^{q_{12}-j-k}$$

$$\left.\left.(1/c_7)^{q_{12}-j-k+q_3+1}\ \Gamma(q_{12}-j-k+q_3+1)-(q_{12}-j)!\ 2^{q_3+1}\ \Gamma(q_3+1)\right]\right\}\right] ; \tag{3.7}$$

where the constants are given by $t^*=r/t^2$, $c_1=a_1'\lambda_1+\lambda_2$, $c_2=a_2'\lambda_1+\lambda_2$, $c_3=\frac{1}{2}(1-\lambda_2/\lambda_1)$, $c_4=c_3/c_1+1/(2\lambda_1)$, $c_5=c_3/c_2+1/(2\lambda_1)$, $c_6=t^*c_4+\frac{1}{2}$, and $c_7=t^*c_5+\frac{1}{2}$.

When R=1 then $\lambda_1=\lambda_2$ and part of the exponential term in (3.6) drops out. By using the procedure leading to (2.11), $P_1(1,\lambda)$ integrates to

$$P_1(1,\lambda) \doteq -k^*\lambda_1^{-\frac{1}{2}f_1}(1/c_8)^{q_{12}+2} \sum_{i=0}^{q_1}\binom{q_1}{i}(-\lambda_2)^{q_1-i}$$

$$[1/(q_{12}-i+1)]\left[(1/c_1)^{q_{12}-i+1}-(1/c_2)^{q_{12}-i+1}\right]$$

$$\left[\sum_{j=0}^{q_{12}+1}\frac{(q_{12}+1)!}{(q_{12}-j+1)!}(t^*c_8)^{q_{12}-j+1}(1/c_9)^{q_{12}-j+q_3+2}\right.$$

$$\left.\Gamma(q_{12}-j+q_3+2)-(q_{12}+1)!\ 2^{q_3+1}\ \Gamma(q_3+1)\right] , \tag{3.8}$$

where $c_8=1/(2\lambda_1)$ and $c_9=t^*c_8+\frac{1}{2}$. This formula yields the same results as the simpler expression (3.14) derived in Section B-3. (3.7) and (3.8) reduce to (2.10) and (2.11), respectively, if we equivalence λ_1 to ν_1/c, λ_2 to ν_2/c, t to t_0, and let $a_1=0$, $a_2=\infty$. This is seen by comparing A_2 to A_1. (3.7) and (3.8) are generalizations of (2.10) and (2.11) in the sense that an option of making a preliminary test is included. They are also generalizations of the work of McCullough (1961) whose formulas included the PTS but were developed for the specific cases studied, with no general derivation attempted.

2. The second component of power

The second component of power as defined in (3.4) can be written as

$$P_2(R,\lambda) = P_{21}(R,\lambda) + P_{22}(R,\lambda),$$

where

$$P_{21}(R,\lambda) \doteq \Pr[X_1/X_2 \le a_1', \frac{crX_3}{\nu_1X_1+\nu_2X_2} > \phi^2(X_1,X_2,\alpha_2)],$$

$$P_{22}(R,\lambda) \doteq \Pr[X_1/X_2 > a_2', \frac{crX_3}{\nu_1X_1+\nu_2X_2} > \phi^2(X_1,X_2,\alpha_2)].$$

The integral expressions for $P_{21}(R,\lambda)$ and $P_{22}(R,\lambda)$ are

$$P_{21}(R,\lambda) \doteq \iiint_{B_2} f(x_1,x_2,x_3)\; dx_1dx_2dx_3,$$

$$P_{22}(R,\lambda) \doteq \iiint_{C_2} f(x_1,x_2,x_3)\; dx_1dx_2dx_3, \tag{3.9}$$

where

$$B_2 = \{(x_1,x_2,x_3):\; x_1 \le a_1'x_2,\; crx_3(\nu_1x_1+\nu_2x_2)$$

$$> (\nu_1t_1x_1+\nu_2t_2x_2)^2,\; x_i>0,\quad i=1,2,3\},$$

$$C_2 = \{(x_1,x_2,x_3): x_1>a_2'x_2,\ crx_3(\nu_1x_1+\nu_2x_2)$$

$$> (\nu_1t_1x_1+\nu_2t_2x_2)^2,\ x_i>0,\quad i=1,2,3,\}.$$

a. The case $n_1=n_2=n$ For this case $t_1=t_2=t_0$ and expressions (3.7) and (3.8) can be used to evaluate the size and approximate power for both $P_{21}(R,\lambda)$ and $P_{22}(R,\lambda)$. The derivation of this is the same as in Chapter II-A-1. We only need to replace a_2' by a_1', a_1' by 0, λ_1 by ν_1/c, λ_2 by ν_2/c, and t by t_0, in A_2, then (3.7) and (3.8) will approximate $P_{21}(R,\lambda)$ for $R\neq1$ and $R=1$, respectively. Similarly, by replacing a_1' by a_2', a_2' by ∞, with the other changes as above, (3.7) and (3.8) will yield $P_{22}(R,\lambda)$ for $R\neq1$ and $R=1$, respectively. (3.15), which will be discussed in Section B-3, is a simpler formula which can be used for this case when R=1.

b. The case $n_1\neq n_2$ Now we will extend the result in (2.15) to a test procedure incorporating a PTS. To evaluate $P_{21}(R,\lambda)$ and $P_{22}(R,\lambda)$ in (3.9) we make the transformation

$$y_1=\nu_1t_1x_1/x_2+\nu_2t_2,\ y_2=x_2,\ y_3=x_3.$$

Then $x_1=y_2(y_1-\nu_2t_2)/(\nu_1t_1)$, $x_2=y_2$, $x_3=y_3$. The Jacobian is $y_2/(\nu_1t_1)$ and we have

$$B_2 = \{(y_1,y_2,y_3):\ 0<y_2<g(y_1)y_3,\ \nu_2t_2<y_1\leq c_{10},\ y_3>0\},$$

$$C_2 = \{(y_1,y_2,y_3):\ 0<y_2<g(y_1)y_3,\ y_1>c_{11},\ y_3>0\},$$

where

$$c_{10}=a_1'\nu_1 t_1+\nu_2 t_2, \qquad c_{11}=a_2'\nu_1 t_1+\nu_2 t_2,$$

and $g(y_1)$ is given in (2.12). The two integrals become

$$P_{21}(R,\lambda) \doteq \int_{\nu_2 t_2}^{c_{10}} \int_0^{\infty} \int_0^{g(y_1)y_3} I' \quad dy_2 dy_3 dy_1,$$

$$P_{22}(R,\lambda) \doteq \int_{c_{11}}^{\infty} \int_0^{\infty} \int_0^{g(y_1)y_3} I' \quad dy_2 dy_3 dy_1,$$

where I' is the same integrand as in (2.13). Also note the first two integrals in (2.13) are like in the above so the second components of power can be written in the same form as (2.15) but with different limits of integration. Therefore

$$P_{21}(R,\lambda) \doteq \int_{\nu_2 t_2}^{c_{10}} I^* \, dy_1, \qquad P_{22}(R,\lambda) = \int_{c_{11}}^{\infty} I^* \, dy_1,$$

where

$$I^* = -[k^*/(\nu_1 t_1)][1/h(y_1)]^{q_{12}+2}[(y_1-\nu_2 t_2)/(\nu_1 t_1)]^{q_1}$$

$$\left\{\sum_{i=0}^{q_{12}+1} \frac{(q_{12}+1)!}{(q_{12}-i+1)!} [g(y_1)h(y_1)]^{q_{12}-i+1} [1/p(y_1)]^{q_{12}-i+q_3+2}\right.$$

$$\Gamma(q_{12}-i+q_3+2)-(q_{12}+1)!\ 2^{q_3+1}\ \Gamma(q_3+1)\Big\}\ .$$

3. Formulas for special cases

In the previous two sections we have shown how numerical computations can be carried out to compute $P_1(R,\lambda)$ and $P_2(R,\lambda)$ and ultimately $P_X(R,\lambda) = P_1(R,\lambda) + P_2(R,\lambda)$. Although the general formulas are complicated, for particular values of n_1, n_2, α_0 and R, the foregoing power formulas reduce to simpler forms, and values for the size and exact power can be easily computed. The following expressions and identities are used to check the validity of the previous derivations of the approximate power formulas. They are also helpful in comparing the approximation of Patnaik (1949) to the exact power for particular cases.

The following power expressions reduce to the size by setting $\lambda=0$, and converting $\chi'^2(1,\lambda)$ to $\chi^2(1)$ and $t'^2(m,\lambda^{\frac{1}{2}})$ to $t^2(m)$. Some of these results are found in McCullough (1961). From (3.1) and (3.2) the following properties are obtained for $\alpha_0=0$, $0<\alpha_0<1$, and $\alpha_0=1$.

a. The case $\alpha_0=0$ This implies $a_1=0$, $a_2=\infty$, and $P_2(R,\lambda)=0$. Hence, the t-test is always used in the final test and $P_X(R,\lambda) = P_1(R,\lambda)$. Also,

$$\lim_{R\to 0} \lambda_1=0, \qquad \lim_{R\to 0} \lambda_2=\frac{n_1+n_2}{n_1(f_1+f_2)}\ ,$$

and

$$\lim_{R\to 0} P_1(R,\lambda) = \Pr\left[\frac{\chi'^2(1,\lambda)}{X_2}\,\frac{n_1(f_1+f_2)}{n_1+n_2} > t^2\right]$$

$$= \Pr\left[t'^2(f_2,\lambda^{\frac{1}{2}}) > \frac{f_2(n_1+n_2)t^2}{n_1(f_1+f_2)}\right]. \tag{3.10}$$

Similarly,

$$\lim_{R \to \infty} \lambda_1 = \frac{n_1+n_2}{n_2(f_1+f_2)}, \qquad \lim_{R \to \infty} \lambda_2=0,$$

and therefore

$$\lim_{R \to \infty} P_1(R,\lambda) = \Pr\left[\frac{\chi'^2(1,\lambda)}{X_1/f_1} > \frac{f_1(n_1+n_2)t^2}{n_2(f_1+f_2)}\right]$$

$$= \Pr\left[t'^2(f_1,\lambda^{\frac{1}{2}}) > \frac{f_1(n_1+n_2)t^2}{n_2(f_1+f_2)}\right]. \tag{3.11}$$

If R=1 then $\lambda_1=\lambda_2=1/(f_1+f_2)$ and

$$P_1(1,\lambda) = \Pr\left[\frac{\chi'^2(1,\lambda)}{\chi^2(f_1+f_2)/(f_1+f_2)} > t^2\right]$$

$$= \Pr[t'^2(f_1+f_2,\lambda^{\frac{1}{2}}) > t^2] \quad (=\alpha_1 \text{ if } \lambda=0).$$

These results hold for both the unilateral and bilateral testing situations, although in the unilateral case, $R<1$ is of no consequence.

b. <u>The case $0<\alpha_0<1$</u> It can be seen that

$$\lim_{R \to 0} P_1(R,\lambda)=0, \qquad \lim_{R \to 0} \nu_1=0, \qquad \lim_{R \to 0} c=1/n_2,$$

$$\lim_{R \to 0} \phi^2(X_1,X_2,\alpha_2)=t_2^2, \qquad \lim_{R \to 0} P_X(R,\lambda)= \lim_{R \to 0} P_2(R,\lambda)$$

$$= \Pr\left[\frac{\chi'^2(1,\lambda)}{X_2/f_2} > t_2^2\right] = \Pr[t'^2(f_2,\lambda^{\frac{1}{2}}) > t_2^2] \quad (=\alpha_2 \text{ for } \lambda=0). \tag{3.12}$$

Further,

$$\lim_{R\to\infty} P_1(R,\lambda)=0, \qquad \lim_{R\to\infty} \nu_1=\infty, \qquad \lim_{R\to\infty} c=\infty,$$

$$\lim_{R\to\infty} \phi^2(X_1,X_2,\alpha_2)=t_1^2, \qquad \lim_{R\to\infty} P_X(R,\lambda)= \lim_{R\to\infty} P_2(R,\lambda)$$

$$= \Pr[t'^2(f_1,\lambda^{\frac{1}{2}}) > t_1^2] \quad (=\alpha_2 \text{ for } \lambda=0). \tag{3.13}$$

When R=1, McCullough (1961) proved that the distributions of u^2 and X_1/X_2 are independent. Therefore

$$P_1(1,\lambda) = \Pr[a_1'<X_1/X_2\leq a_2'] \Pr[t'^2(f_1+f_2,\lambda^{\frac{1}{2}}) > t^2]$$

$$= (1-\alpha_0) \Pr[t'^2(f_1+f_2,\lambda^{\frac{1}{2}}) > t^2] \quad (=(1-\alpha_0)\alpha_1 \text{ if } \lambda=0), \tag{3.14}$$

for both the unilateral and bilateral cases. In addition, if $n_1=n_2=n$ then v^2 and X_1/X_2 are independent and since $t_1=t_2=t_0$ we obtain

$$P_2(1,\lambda) = \alpha_0 \Pr[t'^2(2(n-1),\lambda^{\frac{1}{2}}) > t_0^2]. \tag{3.15}$$

c. <u>The case $\alpha_0=1$</u> Since $a_1=a_2$ in this case, the CT is always used in the final test and $P_X(R,\lambda) = P_2(R,\lambda)$. It can be seen that (3.12), (3.13), (3.14), and (3.15) also hold for this case.

Another property is also noted here. As with the CT made singly, symmetry in the PTS procedure is also present. We find

$$P_X(R,\lambda) \mid (n_1,n_2) = P_X(1/R,\lambda) \mid (n_2,n_1).$$

This property allows us to abbreviate computations as with the single CT of Chapter II.

C. Optimal Preliminary Testing Procedures

So far in this chapter we have defined the general PTS procedure and derived formulas for carrying out computations of the size and power. It remains to discuss the different specific procedures which result from the selection of α_0, α_1, and α_2, to give the test procedure certain desirable properties.

1. Selection of α_0 and α_1 with $\alpha_2=\alpha$ fixed

a. The bilateral case McCullough (1961) and Gurland and McCullough (1962) defined a PTS procedure which uses a t-test to test H_{10} if H_{00} is accepted and a weighted t-test if H_{00} is rejected. H_{00} and H_{10} are defined in (1.2). The final test of H_{10} for this test procedure, denoted by Y_1, has critical region

$$Y_1: \begin{cases} k_1u^2>t^2 & \text{if} \quad 0\le\Sigma_1/\Sigma_2\le a_1 \\ u^2>t^2 & \text{if} \quad a_1<\Sigma_1/\Sigma_2\le a_2 \\ k_2u^2>t^2 & \text{if} \quad a_2<\Sigma_1/\Sigma_2, \end{cases}$$

where k_1, k_2 and t are constants chosen such that $S_X(0)=S_X(1)=S_X(\infty)=\alpha$. We fix $\alpha_2=\alpha$. Let $R\to 0,\infty$ then the PTS approaches an "always reject H_{00}" test and the weighted t-test is always used in the final test. At the extremities in R, the weighted t-test behaves similarly to the standard t-test so from (3.10) and (3.11) it is required that

$$\Pr\left[k_1 t^2(f_2) > \frac{f_2(n_1+n_2)t^2}{n_1(f_1+f_2)}\right] = \alpha_2 = \alpha,$$

$$\Pr\left[k_2 t^2(f_1) > \frac{f_1(n_1+n_2)t^2}{n_2(f_1+f_2)}\right] = \alpha_2 = \alpha.$$

Then

$$k_1 = \frac{f_2(n_1+n_2)t^2}{n_1(f_1+f_2)t_2^2}, \qquad k_2 = \frac{f_1(n_1+n_2)t^2}{n_2(f_1+f_2)t_1^2}\ , \tag{3.16}$$

and the condition $S_X(0)=S_X(\infty) = \alpha$ is satisfied. Now a level for the PTS, α_0, is selected and α_1 is subsequently chosen such that $S_X(1)=\alpha$. Thus t is specified and k_1 and k_2 can be determined from (3.16). The procedure is completely defined now for the particular α_0 selected and $S_X(R)$ can be computed for $0<R<\infty$.

The preliminary testing procedure incorporating a CT as the Behrens-Fisher final test, denoted by Y_2, has critical region

$$Y_2:\begin{cases} u^2>t^2 & \text{if } a_1<\Sigma_1/\Sigma_2\leq a_2 \\ v^2>\left[\dfrac{w_1t_1+w_2t_2}{w_1+w_2}\right]^2 & \text{if } \Sigma_1/\Sigma_2\leq a_1 \quad \text{or } \Sigma_1/\Sigma_2>a_2. \end{cases}$$

As $R \to 0,\infty$ the testing procedure approaches a CT, and from (3.12) and (3.13) the requirement that size equals α at $R=0,\infty$ is automatically satisfied if $\alpha_2=\alpha$. As with Y_1, α_1 is designated after α_0 is selected and hence t is determined to assure that $S_X(1)=\alpha$.

It remains to define a criterion for the selection of an optimal α_0. McCullough (1961) defined two such criteria which are as follows.

i) Conservative criterion: choose that α_0, say α_0^*, such that $S_X(R) \leq \alpha \; \forall R$ and the maximum deviation below α is less than that for any other α_0. In other words, α_0^* is chosen to give the best size α test.

ii) Balanced criterion: choose that α_0, say α_0^*, such that the maximum deviation of $S_X(R)$ above α is equal to the maximum deviation below α over the range of R.

Since the CT and WT were found to be conservative tests, we will be concerned with the conservative criterion for choosing α_0 for the PTS. This will enable us to make valid comparisons between the PTS procedures and the CT and WT made singly.

b. <u>The unilateral case</u> For the unilateral case, Gurland and McCullough (1962) defined two types of PTS procedures. The first is simply a unilateral version of Y_1. It is given by

$$Y_1': \begin{cases} u^2 > t^2 & \text{if} \quad 0 \leq \Sigma_1/\Sigma_2 \leq a_0 \\ ku^2 > t^2 & \text{if} \quad a_0 < \Sigma_1/\Sigma_2 . \end{cases}$$

In a manner similar to the bilateral test, k and t are chosen after α_0 is specified to assure $S_X(1) = S_X(\infty) = \alpha$. It is found that

$$k = \frac{f_1(n_1+n_2)t^2}{n_2(f_1+f_2)t_1^2} .$$

Then after α_0 is chosen α_1 is selected and hence t determined such that $S_X(1) = \alpha$.

The second procedure discussed by Gurland and McCullough (1962) incorporates the unilateral WT. It has critical region

$$Y_2': \begin{cases} u^2 > t^2 & \text{if} \quad 0 \le \Sigma_1/\Sigma_2 \le a_0 \\ Y(r_1, r_2) > 1 & \text{if} \quad a_0 < \Sigma_1/\Sigma_2, \end{cases}$$

where $Y(r_1, r_2)$ is given in (1.8), (1.9) and r_2 and r_1 are chosen so that the size equals α at $R=1,\infty$, respectively. It is found r_1 is the same as for the bilateral case, i.e.,

$$r_1 = t_1^2/(n_1 f_1).$$

r_2 is the same as r_2' in the unilateral WT which was discussed in Chapter II-C. t is chosen as in Y_1'.

Another procedure we wish to consider is the unilateral counterpart of Y_2. It has critical region

$$Y_3': \begin{cases} u^2 > t^2 & \text{if} \quad 0 \le \Sigma_1/\Sigma_2 \le a_0 \\ v^2 > \left[\dfrac{w_1 t_1 + w_2 t_2}{w_1 + w_2}\right]^2 & \text{if} \quad a_0 < \Sigma_1/\Sigma_2. \end{cases}$$

Unlike Y_2', for Y_3' we choose to use t_1 and t_2 as defined for the bilateral test rather than t_1 and t_2' for a unilateral CT as described in Chapter II-C, even though we are only concerned with the range $R \ge 1$.

The above three test procedures are somewhat similar in nature, hence a comparison is made. A study of their behavior in a theoretical sense is feasible only as $R \to 1,\infty$. This was discussed in Section B-3. In order to determine the general behavior for $1<R<\infty$, an empirical study of several small samples was carried out using the derived computational formulas of Section B. As mentioned previously the conservative criterion is used to facilitate comparisons with the conservative tests of Chapter II. We will use $\alpha_2=\alpha=.05$.

Tables 13, 14, and 15 specify the constants needed to completely define procedures Y_1', Y_2', Y_3' for six particular cases. Tables 16, 17, and 18 show how size varies over R for the three procedures. It can be seen that Y_1' has the best size control for $n_1>n_2$ and Y_2' appears to have less size fluctuations for $n_1<n_2$. Y_3' does not control the size quite as well as Y_1' and Y_2' in most instances. Size control is better with Y_2' than with the single WT.

2. Selection of α_0 with $\alpha_1=\alpha_2=\alpha$ fixed

It is seen that gains can be made by using PTS procedures of the previous section as opposed to using a single test. However, the procedures suffer from a disadvantage which limits their usefulness. The statistic used in the final test depends on the outcome of the PTS, as we would expect. But furthermore, the level α_1 depends on the level of the PTS, α_0. Hence computations are made more complicated and much greater reliability is placed on one final test statistic than the other. It was also discovered empirically that for many cases, there does not exist an α_0^* satisfying the conservative criterion for the bilateral procedure Y_2 and the bilateral counterpart of Y_2'. α_0^* values for the balanced criterion do not always exist either. Thus, we are prompted to explore a simpler procedure which has more intuitive appeal.

It appears that a natural and practical way to handle the PTS problem (1.2) would be to make a preliminary test at some level, followed by the final test at another level which is pre-specified. In other words, the preliminary test dictates only the statistic to be used in the final test and not the level. We will investigate the bilateral procedure, Y_2, to see if gains in size and power can be made relative to single test procedures discussed in Chapter II.

Let $\alpha_1=\alpha_2=\alpha=.05$. Then using procedure Y_2 for several different values of α_0, α_0^* can be chosen according to the conservative criterion. The constants specifying procedure Y_2 are given in Table 19 for several cases. The size and power values for these cases are tabulated in Tables 20 and 21.

Some interesting relationships can be seen from these results. When $n_1=n_2=n$, and n is increasing, α_0^* decreases to as small as .03 when n=13. This indicates the PTS procedure is essentially a t-test in this instance. At the other extreme, $\alpha_0^*=.98$ for $n_1=3$, $n_2=13$. Here, the PTS procedure is essentially a single CT.

For the bilateral case the α_0^* corresponding to (n_1,n_2) is the same as that for (n_2,n_1). It is apparent that as $|n_1-n_2|$ increases and min (n_1,n_2) decreases, α_0^* tends to become large. This is seen by

$\lvert n_1-n_2\rvert$	$\min(n_1,n_2)$	α_0^*
4	5	.55
4	3	.88
10	3	.98

The real value of a PTS procedure with $\alpha_1=\alpha_2=\alpha$ can be seen by comparing Tables 3 and 20. For α_0^*, the procedure Y_2 outlined above has uniformly better size control than when the CT is used alone. The gains in power are evident upon examining Table 21 versus Tables 8, 9, and 10. Substantial improvement is made in some cases, particularly when $n_1=n_2=n$. As a general scheme for approximating α_0^* for equal sample sizes, consider using

$$\alpha_0^* = 5/n^2.$$

This approximate value of α_0^* tends to be a little large, hence the resulting tests will be slightly more conservative than the tests using the α_0^* values given in Table 19. This approximation fits the empirical results acceptably for $n>3$. For $n=3$, we recommend to use $\alpha_0^*=.47$.

IV. THE SIZE AND POWER OF THE MULTIVARIATE COCHRAN'S TEST

Suppose a sample of size n_1, $\underline{x}_{11}$, $\underline{x}_{12}$, $\underline{x}_{13}$, ... , $\underline{x}_{1n_1}$, is obtained from a p-variate normal, $N_p(\underline{\mu}_1, \Sigma_1)$. A second independent random sample of size n_2, $\underline{x}_{21}$, $\underline{x}_{22}$, $\underline{x}_{23}$, ... , $\underline{x}_{2n_2}$, is available from $N_p(\underline{\mu}_2, \Sigma_2)$, where $\underline{x}_{ij}$, and $\underline{\mu}_i$ are pxl vectors and Σ_i are pxp positive definite matrices for i=1,2. It is desired to make a test of the hypothesis $H_0:\underline{\mu}_1=\underline{\mu}_2$ against the alternative $H_1:\underline{\mu}_1\neq\underline{\mu}_2$. When $\Sigma_1\neq\Sigma_2$ we have the multivariate Behrens-Fisher problem. The test used is what we will call the multivariate Cochran's test (MCT), which has a critical region

$$V^2 = (\overline{\underline{x}}_1-\overline{\underline{x}}_2)'S^{-1}(\overline{\underline{x}}_1-\overline{\underline{x}}_2) > \left[\frac{W_1T_1+W_2T_2}{W_1+W_2}\right]^2, \tag{4.1}$$

where S, W_1, W_2, T_1, and T_2 are given in (1.6). As in previous chapters we denote $f_i=n_i-1$, i=1,2. The distributions of the components comprising this test statistic are

$$\overline{\underline{x}}_1-\overline{\underline{x}}_2 \sim N_p(\underline{\mu}_1-\underline{\mu}_2, \Sigma_1/n_1+\Sigma_2/n_2),$$

$$f_iS_i \sim W(f_i, \Sigma_i),$$

$$W_i \sim [|\Sigma_i|/(n_i-1)^p]\ \chi^2(n_i-1)\ \chi^2(n_i-2)\ \dots\ \chi^2(n_i-p),\ i=1,2.$$

$W(f_i, \Sigma_i)$ is the Wishart distribution with f_i degrees of freedom and covariance matrix, Σ_i. Unfortunately, the quantity

$$S = S_1/n_1+S_2/n_2$$

is not a Wishart variate in general because S_1/n_1 and S_2/n_2 are unbiased estimates of different covariance matrices, Σ_1/n_1 and Σ_2/n_2, respectively.

However, if it happens that

$$\Sigma_1/(n_1f_1) = \Sigma_2/(n_2f_2) = \Sigma_0,$$

then V^2 is a T^2 variate since

$$S_i/n_i \sim W(f_i, \Sigma_0),\ i=1,2.$$

So for this case

$$S_1/n_1+S_2/n_2 \sim W(f_1+f_2, \Sigma_0),$$

and a test with the critical region given by

$$V^2 > T^2_\alpha(p, f_1+f_2-p+1)$$

is an exact level α test of H_0. This result will be used later to check the approximate procedure. For general Σ_1 and Σ_2 though, the distribution of the MCT statistic is very complicated. An adequate size and power study can be made for the bivariate case using Monte Carlo techniques as developed in Yao (1962, 1965). The rest of this chapter will be concerned with the development of techniques to carry out computations for the selected cases $(f_1,f_2) = (6,6)$, $(6,12)$, $(6,18)$, and $(12,12)$. Then the size and power for these cases will be examined and compared to the results of Yao.

A. Probability Expression for the Power Function

In the univariate problem, we were confronted with two nuisance parameters, σ_1^2 and σ_2^2 which were converted to one parameter, $R=\sigma_1^2/\sigma_2^2$. Similarly we shall first express Σ_1 and Σ_2 in terms of a single matrix, say Λ. It is known that there exists a non-singular matrix, C, such that

$$C(\Sigma_1/n_1)C' = \Lambda,$$

$$C(\Sigma_2/n_2)C' = I-\Lambda,$$

where I is the identity matrix and Λ is a diagonal matrix with elements, $0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_p \le 1$. Let

$$\underline{z} = C(\overline{\underline{x}}_1 - \overline{\underline{x}}_2).$$

Then the critical region in (4.1) becomes

$$V^2 = \underline{z}'C^{-1'}(S_1/n_1+S_2/n_2)^{-1}C^{-1}\underline{z} > \left[\frac{W_1T_1+W_2T_2}{W_1+W_2}\right]^2.$$

Let

$$M_i = C(S_i/n_i)C' \quad i=1,2.$$

Then

$$V^2 = \underline{z}'C^{-1'}(C^{-1}M_1C'^{-1}+C^{-1}M_2C'^{-1})^{-1}C^{-1}\underline{z}$$

$$= \underline{z}'C^{-1'}C'(M_1+M_2)^{-1}CC^{-1}\underline{z}$$

$$= \underline{z}'M^{-1}\underline{z},$$

where $M=M_1+M_2$. The quantity on the right hand side of (4.1) can be transformed by multiplying the numerator and denominator by $|C||C'|$. We can write

$$|C||C'|\ W_i = |C|W_i|C'| = |C(S_i/n_i)C'| = |M_i|, \quad i=1,2.$$

Thus (4.1) can be written in the equivalent form

$$V^2 = \underline{z}'M^{-1}\underline{z} > G^2(M_1,M_2,\alpha),$$

where

$$G(M_1,M_2,\alpha) = \frac{|M_1|T_1+|M_2|T_2}{|M_1| + |M_2|}.$$

The distributions involved are

$$\underline{z} \sim N_p(\underline{\delta},I),$$

$$M_1 \sim W(f_1,\Lambda/f_1),$$

$$M_2 \sim W(f_2,(I-\Lambda)/f_2), \tag{4.2}$$

where

$$\underline{\delta} = C(\underline{\mu}_1-\underline{\mu}_2).$$

Under H_0, $\underline{z} \sim N(\underline{0},I)$ so the size of the MCT is

$$S_C(\Lambda) = Pr[\underline{z}'M^{-1}\underline{z} > G^2(M_1,M_2,\alpha) \mid \underline{\delta}=\underline{0}]. \tag{4.3}$$

$S_C(\Lambda)$ depends on f_1, f_2, α, and Λ. The power of the test can be written in a similar fashion as

$$P_C(\Lambda,\theta) = Pr[\underline{z}'M^{-1}\underline{z} > G^2(M_1,M_2,\alpha) \mid \underline{\delta}], \tag{4.4}$$

where

$$\theta = \underline{\delta}'\underline{\delta}.$$

Also, the size can be expressed in terms of power by the relationship

$$S_C(\Lambda) = P_C(\Lambda,0).$$

With this in mind, we will concern ourselves primarily with the development of $P_C(\Lambda,\theta)$.

B. Methods for Computing the Size and Power

1. Generation of random samples

As mentioned earlier, a Monte Carlo study of size and power will be made for the bivariate Cochran's test for selected cases. Given f_1, f_2, α, and Λ, we generate N sets of the random matrices, $\underline{z}$, M_1, and M_2 having the desired distributions, (4.2), and then estimate $S_C(\Lambda)$ and $P_C(\Lambda,\theta)$ on the bases of these samples. Generation of normal deviates can be carried out using any one of several techniques. We will use the power residue method to obtain uniform random numbers and then the method of Box and Muller (1958) to obtain the normal random numbers.

The random matrices M_1 and M_2 can be generated using a Bartlett decomposition. We have

$$W(f,I) = \begin{bmatrix} y_1^2 & y_1x_2 & y_1x_3 & \cdots & y_1x_p \\ y_1x_2 & y_2^2+x_2^2 & y_2x_3 & \cdots & y_2x_p \\ y_1x_3 & y_2x_3 & y_2^2+\sum_{j=2}^{3}x_j^2 & \cdots & y_3x_p \\ \vdots & \vdots & \vdots & \ddots & \\ y_1x_p & y_2x_p & y_3x_p & \cdots & y_2^2+\sum_{j=2}^{p}x_j^2 \end{bmatrix},$$

where each

$$y_i^2 \sim \chi^2(f-i+1) \qquad i=1,2,3, \ldots , p-1,$$

$$x_j \sim N(0,1) \qquad j=2,3,4, \ldots , p,$$

and all y_i and x_j are independent.

Since

$$\chi^2(f-i+1) = \sum_{j=1}^{f-i+1} x_j^2,$$

the method of Box and Muller (1958) can also be used to generate the Wishart matrices.

It is desirable to use the same samples in as many computations as possible so that differences encountered in comparing results are due to the differences in distribution rather than random variation. Consequently we will use the same samples for all different Λ specified to reflect size and power variations due to Λ. The same random Wishart matrices can be used to an extent for computing M_1 and M_2 as f_1 and f_2 change. We will be concerned with $(f_1,f_2) = (6,6)$, $(6,12)$, $(6,18)$, and $(12,12)$, so for each of the N random samples we generate four independent matrices, each with distribution $W(6,I)$. Denote these four matrices by Z_1, Z_2, Z_3, Z_4. Then we can find the appropriate Wishart matrices by combining the Z_i in the following manner.

(f_1,f_2)	$W(f_1,I)$	$W(f_2,I)$
(6,6)	Z_1	Z_4
(6,12)	Z_1	Z_3+Z_4
(6,18)	Z_1	$Z_2+Z_3+Z_4$
(12,12)	Z_1+Z_2	Z_3+Z_4

Now let

$$M_1 = (\Lambda^{\frac{1}{2}}W(f_1,I)\Lambda^{\frac{1}{2}})/f_1,$$

$$M_2 = ((I-\Lambda)^{\frac{1}{2}}W(f_2,I)(I-\Lambda)^{\frac{1}{2}})/f_2.$$

Then

$$M_1 \sim W(f_1, \Lambda/f_1),$$

$$M_2 \sim W(f_2, (I-\Lambda)/f_2),$$

which are the desired distributions.

2. Monte Carlo estimation

The estimation of $S_C(\Lambda)$ and $P_C(\Lambda,\theta)$, once the random sampling has been completed, can be accomplished in different ways. One way is the straight relative frequency approach. That is, for specified Λ and θ

$$P_C(\Lambda,\theta) = n_0/N,$$

where n_0 is the number of samples such that $\underline{z}'M^{-1}\underline{z} > G^2(M_1,M_2,\alpha)$. This method, however, is inferior to the method below which expresses (4.3) and (4.4) in terms of the F-distribution.

In this connection, denote the $(i,j)^{th}$ element of the pxp Wishart matrix, M_1, by m_{ij}. It is known that

$$f_1 m_{ii}/\lambda_i \sim \chi^2(f_1), \quad i=1,2,3, \ldots , p,$$

and that the diagonal elements of M_1 are independent. Therefore,

$$\mathrm{tr}[f_1\Lambda^{-\frac{1}{2}}M_1\Lambda^{-\frac{1}{2}}] \sim \chi^2(pf_1)$$

and hence

$$w = \mathrm{tr}[f_1\Lambda^{-\frac{1}{2}}M_1\Lambda^{-\frac{1}{2}}+f_2(I-\Lambda)^{-\frac{1}{2}}M_2(I-\Lambda)^{-\frac{1}{2}}] \sim \chi^2(p(f_1+f_2)).$$

w is independent of M_1/w and M_2/w (see Yao, 1962). w can also be written in the equivalent form

$$w = \mathrm{tr}[W(f_1,I)+W(f_2,I)],$$

so the Wishart matrices generated by the method of Section B-1 can be used directly. Under H_0, $\underline{z} \sim N_p(\underline{0},I)$ and $\underline{z}'\underline{z} \sim \chi^2(p)$. $\underline{z}'\underline{z}$ is seen to be independent of $\underline{d} = \underline{z}/(\underline{z}'\underline{z})^{\frac{1}{2}}$.

Let

$$F(p,p(f_1+f_2)) = \frac{(\underline{z}'\underline{z})/p}{w/(p(f_1+f_2))} .$$

Then $F(p,p(f_1+f_2))$ is a variate from the F-distribution with $(p,p(f_1+f_2))$ degrees of freedom. $F(p,p(f_1+f_2))$ is independent of $\underline{d}$, M_1/w, and M_2/w. Since

$$\underline{z}'M^{-1}\underline{z} = (\underline{z}'\underline{z}/w)(\underline{d}'(M/w)^{-1}\underline{d}),$$

the size of the MCT, from (4.3) for a given set $(f_1, f_2, \alpha, \Lambda)$ and fixed i^{th} sample $(\underline{d}, M_1, M_2)_i$, can be expressed as

$$S_{Ci}(\Lambda) = \Pr[F(p,p(f_1+f_2)) > \frac{(f_1+f_2)G^2(M_1,M_2,\alpha)}{\underline{d}'(M/w)^{-1}\underline{d}} \mid (\underline{d},M_1,M_2)_i].$$

It is noted that $S_{Ci}(\Lambda)$ is the same as the result of Yao (1962, 1965) if we let

$$G^2(M_1,M_2,\alpha) = T^2_\alpha(p,f_T-p+1).$$

In a similar manner, let $\underline{z}^* \sim N_p(\underline{\delta},I)$ when $\underline{\delta}\neq\underline{0}$. Then $\underline{z}^{*\prime}\underline{z}^* \sim \chi'^2(p,\theta)$, independently of $\underline{d}^* = \underline{z}^*/(\underline{z}^{*\prime}\underline{z}^*)^{\frac{1}{2}}$. It is evident that

$$F'(p,p(f_1+f_2),\theta) = \frac{(\underline{z}^{*\prime}\underline{z}^{*})/p}{w/(p(f_1+f_2))}$$

is a variate from the non-central F-distribution with non-centrality parameter, θ. The power of the MCT, from (4.4) for a given set (f_1, f_2, α, Λ, $\underline{\delta}$) and fixed i^{th} sample, can be written as

$$P_{Ci}(\Lambda,\theta) = \Pr[F'(p,p(f_1+f_2),\theta) > \frac{(f_1+f_2)G^2(M_1,M_2,\alpha)}{\underline{d}^{*\prime}(M/w)^{-1}\underline{d}^*} \mid (\underline{d}^*,M_1,M_2)_i].$$

Define the random variables

$$R_C = \frac{(f_1+f_2)G^2(M_1,M_2,\alpha)}{\underline{d}'(M/w)^{-1}\underline{d}}, \qquad R_C^* = \frac{(f_1+f_2)G^2(M_1,M_2,\alpha)}{\underline{d}^{*\prime}(M/w)^{-1}\underline{d}^*},$$

$$R_E = \frac{(f_1+f_2)T_\alpha^2(p,f_1+f_2-p+1)}{\underline{d}'(M/w)^{-1}\underline{d}}, \qquad R_E^* = \frac{(f_1+f_2)T_\alpha^2(p,f_1+f_2-p+1)}{\underline{d}^{*\prime}(M/w)^{-1}\underline{d}^*}.$$

These random variables take the N values,

$$R_C = r_{Ci},\ R_C^* = r_{Ci}^*,\ R_E = r_{Ei},\ R_E^* = r_{Ei}^*,\quad i=1,2,3,\ \ldots\ ,N.$$

We can write the estimates for the size and power based on sample i by

$$S_{Ci}(\Lambda) = \Pr[F(p,p(f_1+f_2)) > r_{Ci}],$$

$$P_{Ci}(\Lambda,\theta) = \Pr[F'(p,p(f_1+f_2),\theta) > r_{Ci}^*].$$

Since approximations of the size and power are given by each sample, logical estimates of the size and power would be the average of these approximations over the N samples.

$$\overline{S}_C(\Lambda) = \sum_{i=1}^{N} S_{Ci}(\Lambda)/N,$$

$$\overline{P}_C(\Lambda,\theta) = \sum_{i=1}^{N} P_{Ci}(\Lambda,\theta)/N.$$

In the rest of this section we shall develop an improvement of this estimate. It was pointed out at the beginning of this chapter that when

$$\Sigma_1/(n_1 f_1) = \Sigma_2/(n_2 f_2),$$

we have an exact level α test and

$$S_E(\Lambda_E) = \Pr[V^2 > T_\alpha^2(p, f_1+f_2-p+1)\,|\,\underline{\delta}=\underline{0}] = \alpha,$$

$$P_E(\Lambda_E,\theta) = \Pr[V^2 > T_\alpha^2(p, f_1+f_2-p+1)\,|\,\underline{\delta}], \tag{4.5}$$

where

$$\Lambda/f_1 = (I-\Lambda)/f_2,$$

$$\Lambda = \Lambda_E = (f_1/(f_1+f_2))I.$$

The size and power in this instance can be estimated, respectively, by

$$\overline{S}_E(\Lambda_E) = \sum_{i=1}^{N} S_{Ei}(\Lambda_E)/N,$$

$$\overline{P}_E(\Lambda_E,\theta) = \sum_{i=1}^{N} P_{Ei}(\Lambda_E,\theta)/N, \tag{4.6}$$

where

$$S_{Ei}(\Lambda_E) = \Pr[F(p, p(f_1+f_2)) > r_{Ei}],$$

$$P_{Ei}(\Lambda_E,\theta) = \Pr[F'(p, p(f_1+f_2),\theta) > r_{Ei}^*].$$

Regressing $P_{Ci}(\Lambda,\theta)$ on $P_{Ei}(\Lambda_E,\theta)$ as given in Cochran (1963, pp. 193-196), we have

$$p_i = \overline{p} + \beta(q_i-\overline{q}) + e_i, \quad i=1,2,3, \ldots , N,$$

where

$$p_i = P_{Ci}(\Lambda,\theta), \qquad q_i = P_{Ei}(\Lambda_E,\theta),$$

$$\overline{p} = \overline{P}_C(\Lambda,\theta), \qquad \overline{q} = \overline{P}_E(\Lambda_E,\theta),$$

β is an unknown regression coefficient, and the e_i are random errors.

The least squares estimator for β is given by

$$\hat{\beta} = b = \sum_{i=1}^{N} (p_i-\overline{p})(q_i-\overline{q})/\sum_{i=1}^{N} (q_i-\overline{q})^2.$$

Now let

$$\hat{p} = \overline{p} + b(\overline{q}-E(\overline{q})).$$

$E(\overline{q}) = S_E(\Lambda_E) = \alpha$ for the size $(\theta=0)$. For $\theta\neq 0$, $E(\overline{q}) = P_E(\Lambda_E,\theta)$, the power of the F-test with (p,f_1+f_2-p+1) degrees of freedom. $\hat{p}$ is the regression estimate of $P_C(\Lambda,\theta)$ which has a sample variance estimated by

$$\hat{Var}(\hat{p}) = \frac{1}{N(N-2)}\left[\sum_{i=1}^{N}(p_i-\overline{p})^2 - b^2\sum_{i=1}^{N}(q_i-\overline{q})^2\right]$$

for large N. It is seen from Cochran (1963, p. 199) that

$$Var(\hat{p}) \leq Var(\overline{p}),$$

and, therefore, $\hat{p}$ will be used to estimate $P_C(\Lambda,\theta)$ in the computations. The $100(1-\alpha)\%$ confidence interval can be obtained for large samples by

$$\hat{p} \pm [\hat{Var}(\hat{p})]^{\frac{1}{2}}Z_{\frac{1}{2}\alpha}.$$

3. Exact formulas for special cases

It was pointed out in Chapter I-A-2 that the MCT is asymptotically the chi-square test in (1.4). This can be seen as follows. It is easily shown from the density functions of F and χ^2 that

$$\lim_{n_i \to \infty} F_\alpha(p,n_i-1) = \chi^2(p)/p, \quad i=1,2.$$

Also, since S_i is an unbiased and consistent estimate of Σ_i we know that

$$\lim_{n_i \to \infty} S_i = \Sigma_i, \quad i=1,2.$$

Hence (4.1) will reduce to (1.4) for large n_1,n_2 and the size and power of the test are easily computed.

Some behavior patterns of the size and power of the MCT for smaller samples can be recognized by setting $\lambda_1=\lambda_2=\lambda_0$. Then

$$C(\Sigma_1/n_1)C' = \lambda_0 I,$$

$$C(\Sigma_2/n_2)C' = (1-\lambda_0)I,$$

and

$$\Sigma_1/(n_1\lambda_0) = C^{-1}C'^{-1} = \Sigma_1/n_1+\Sigma_2/n_2.$$

Therefore,

$$\Sigma_2 = \frac{n_2(1-\lambda_0)}{n_1\lambda_0}\Sigma_1, \tag{4.7}$$

$$|\Sigma_1|/|\Sigma_2| = \left[\frac{n_1\lambda_0}{n_2(1-\lambda_0)}\right]^p.$$

Then

$\lambda_0 \to 0$ implies $|\Sigma_1|/|\Sigma_2| \to 0$,

$\lambda_0 \to 1$ implies $|\Sigma_1|/|\Sigma_2| \to \infty$.

So λ_0 at its extremes implies a large difference in the covariance matrices Σ_1 and Σ_2. From (4.7)

$\lambda_0 = n_2/(n_1+n_2)$ implies $\Sigma_1=\Sigma_2$.

From (4.1), (4.3), (4.4), and the above results

$$\lambda_0 \to 0 \text{ implies } S_C(\Lambda) = \Pr[(\underline{\bar{x}}_1-\underline{\bar{x}}_2)'(S_2/n_2)^{-1}(\underline{\bar{x}}_1-\underline{\bar{x}}_2) > T_2^2]$$

$$= \Pr[F(p,n_2-p) > F_\alpha(p,n_2-p)] = \alpha,$$

$$\lambda_0 \to 1 \text{ implies } S_C(\Lambda) = \Pr[(\underline{\bar{x}}_1-\underline{\bar{x}}_2)'(S_1/n_1)^{-1}(\underline{\bar{x}}_1-\underline{\bar{x}}_2) > T_1^2]$$

$$= \Pr[F(p,n_1-p) > F_\alpha(p,n_1-p)] = \alpha. \qquad (4.8)$$

Analogously,

$$\lambda_0 \to 0 \text{ implies } P_C(\Lambda,\theta) = \Pr[F'(p,n_2-p,\theta) > F_\alpha(p,n_2-p)],$$

$$\lambda_0 \to 1 \text{ implies } P_C(\Lambda,\theta) = \Pr[F'(p,n_1-p,\theta) > F_\alpha(p,n_1-p)]. \qquad (4.9)$$

Symmetry in the MCT is similar to that in the univariate case. By interchanging f_1 and f_2, Λ and $I-\Lambda$, the same size and power can be obtained. We will take advantage of the symmetry and consider only the cases when $f_1 \leq f_2$.

C. An Empirical Study

In order to see how the size and power behave for particular cases, an empirical study was undertaken for $(f_1,f_2) = (6,6)$, $(6,12)$, $(6,18)$, and

(12,12). We let α=.05 and use N=500 samples. The essential results are presented in Tables 22 through 30.

Table 22 contains a comparison of the approximate Monte Carlo estimates and the exact value for the case where an exact test exists, i.e.,

$$\ddagger_1/(n_1f_1) = \ddagger_2/(n_2f_2).$$

Here we consider the mean estimate, $\overline{P}_E(\Lambda_E,\theta)$, given in (4.6). The exact value, $P_E(\Lambda_E,\theta)$, is found from (4.5) using the tables of Tang (1938). It is seen from Table 22 that the Monte Carlo and the exact values are surprisingly close.

Tables 23 through 26 contain estimates for $S_C(\Lambda)$ using the regression technique of Section B-2. These size computations for the MCT are compared to the APDF test procedure of Yao (1962, 1965) using the same random samples. The special properties in (4.8) are apparent. As $\lambda_0 \to 0,1$, the size approaches α=.05. In a manner similar to the univariate case, the test is conservative and appears to take its minimum size at $\lambda_1=\lambda_2=.50$ for $f_1=f_2$. This is equivalent to $\ddagger_1=\ddagger_2$ from (4.7). When f_2-f_1 is large and λ_1, λ_2 are equal and small, it appears that the size may be slightly greater than α. Again this is consistent with the univariate case. By symmetry, when $f_1=f_2$ we would expect the size for λ_1, λ_2 to be equal to that for $1-\lambda_1$ and $1-\lambda_2$, respectively. It can be seen they are close but differ because of the random sampling involved in the computations.

The behavior of the APDF test is similar to what Yao (1962) claimed. The size of the test appears to deviate from α less than that for the MCT but it is not conservative, hence is not a size α test. Which test is better in a given circumstance depends on the importance of bounding Type I error.

The generally shorter 95% confidence intervals for the size of the MCT indicate that the variance of the estimate of $S_C(\Lambda)$ is slightly less than that for the APDF test.

Tables 27 through 30 contain the estimates of $P_C(\Lambda,\theta)$, the power of the MCT. Although the power is really dependent on $\underline{\delta}$, it was found that $P_C(\Lambda,\theta)$ remains fairly stable for various values of $\underline{\delta}$ as long as $\theta=\underline{\delta}'\underline{\delta}$ is constant. Any variation is reflected in both the MCT and APDF so for comparative purposes, it suffices to examine a single $\underline{\delta}$ vector for each θ we choose to specify. We will let $\underline{\delta}'=(\delta_1,\delta_2)$, where $\delta_1=\delta_2=(\theta/2)^{1/2}$.

In the examination of these tables, the properties in (4.9) are apparent. When $\lambda_1=\lambda_2=\lambda_0$ and $\lambda_0 \to 0,1$, the power approaches the power of the appropriate F-test. For the most part, the power behavior of the MCT is not surprising as it is similar to the behavior of the univariate Cochran's test.

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VII. APPENDIX

Table 1. Size of t_R^2 test, α=.05

n_1	n_2	R=1	R=2	R=4	R=10	R=40	R=100	R=1000
3	3	.0500	.0530	.0613	.0766	.0960	.1028	.1082
3	5	.0500	.0718	.1022	.1453	.1900	.2040	.2145
3	7	.0500	.0862	.1354	.2016	.2663	.2858	.3002
3	13	.0500	.1122	.1970	.3062	.4079	.4381	.4604
5	3	.0500	.0376	.0320	.0300	.0303	.0307	.0310
5	5	.0500	.0524	.0585	.0681	.0776	.0803	.0822
5	9	.0500	.0755	.1069	.1446	.1769	.1854	.1911
7	3	.0500	.0292	.0191	.0137	.0114	.0110	.0108
7	7	.0500	.0519	.0565	.0632	.0693	.0710	.0721
7	13	.0500	.0769	.1081	.1432	.1711	.1781	.1826
9	5	.0500	.0338	.0253	.0204	.0183	.0179	.0177
9	9	.0500	.0515	.0552	.0603	.0648	.0660	.0667
13	3	.0500	.0182	.0064	.0021	.0009	.0007	.0006
13	7	.0500	.0322	.0225	.0169	.0142	.0137	.0134
13	13	.0500	.0511	.0537	.0518	.0601	.0608	.0613

Table 2. Size of Welch-type test, α=.05[a]

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0127	.0171	.0247	.0438	.0500
3	5	.0191	.0302	.0384	.0482	.0500
3	7	.0240	.0362	.0427	.0490	.0500
3	9	.0275	.0391	.0447	.0493	.0500
5	3	.0191	.0178	.0229	.0424	.0500
5	5	.0241	.0301	.0375	.0481	.0500
5	7	.0278	.0362	.0424	.0490	.0500
5	9	.0306	.0396	.0447	.0493	.0500
7	3	.0240	.0189	.0219	.0408	.0500
7	5	.0278	.0300	.0365	.0478	.0500
7	7	.0308	.0361	.0419	.0489	.0500
7	9	.0330	.0395	.0444	.0493	.0500
9	3	.0275	.0201	.0214	.0394	.0500
9	5	.0306	.0302	.0356	.0474	.0500
9	7	.0330	.0360	.0413	.0487	.0500
9	9	.0348	.0394	.0440	.0492	.0500

[a]Source: McCullough (1961, p. 37).

Table 3. Size of Cochran's test, α=.05

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0126	.0171	.0247	.0438	.0500
3	5	.0204	.0317	.0398	.0486	.0500
3	7	.0265	.0388	.0447	.0495	.0500
3	9	.0310	.0425	.0469	.0498	.0500
3	13	.0366	.0461	.0487	.0500	.0500
3	21	.0423	.0485	.0498	.0500	.0500
5	3	.0204	.0190	.0242	.0432	.0500
5	5	.0241	.0301	.0375	.0481	.0500
5	7	.0281	.0364	.0426	.0490	.0500
5	9	.0312	.0401	.0449	.0494	.0500
5	13	.0354	.0436	.0470	.0497	.0500
5	21	.0401	.0465	.0486	.0498	.0500
7	3	.0265	.0210	.0242	.0423	.0500
7	5	.0281	.0302	.0366	.0478	.0500
7	7	.0308	.0361	.0419	.0489	.0500
7	9	.0331	.0397	.0444	.0493	.0500
7	13	.0364	.0432	.0467	.0496	.0500
7	21	.0403	.0461	.0483	.0498	.0500
9	3	.0310	.0229	.0243	.0414	.0500
9	5	.0312	.0306	.0360	.0475	.0500
9	7	.0331	.0361	.0414	.0488	.0500
9	9	.0348	.0394	.0440	.0492	.0500
9	13	.0375	.0430	.0464	.0496	.0500
9	21	.0409	.0460	.0481	.0498	.0500
13	3	.0366	.0261	.0248	.0397	.0500
13	5	.0354	.0316	.0351	.0469	.0500
13	7	.0364	.0362	.0405	.0485	.0500
13	9	.0375	.0393	.0434	.0491	.0500
13	13	.0394	.0428	.0461	.0495	.0500
13	21	.0419	.0458	.0480	.0498	.0500
21	3	.0423	.0309	.0264	.0369	.0500
21	5	.0401	.0336	.0344	.0458	.0500
21	7	.0403	.0370	.0395	.0480	.0500
21	9	.0409	.0395	.0425	.0487	.0500
21	13	.0419	.0427	.0455	.0494	.0500
21	21	.0434	.0456	.0477	.0497	.0500

Table 4. Size of Welch-type test, $\alpha=.01$[a]

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0004	.0009	.0016	.0059	.0100
3	5	.0012	.0030	.0052	.0091	.0100
3	7	.0020	.0048	.0070	.0095	.0100
3	9	.0028	.0060	.0078	.0097	.0100
5	3	.0012	.0010	.0014	.0051	.0100
5	5	.0018	.0028	.0046	.0089	.0100
5	7	.0024	.0045	.0066	.0095	.0100
5	9	.0031	.0057	.0076	.0097	.0100
7	3	.0020	.0011	.0013	.0045	.0100
7	5	.0024	.0026	.0042	.0087	.0100
7	7	.0030	.0043	.0063	.0094	.0100
7	9	.0036	.0055	.0074	.0096	.0100
9	3	.0028	.0013	.0013	.0042	.0100
9	5	.0031	.0028	.0039	.0084	.0100
9	7	.0036	.0043	.0060	.0093	.0100
9	9	.0042	.0054	.0072	.0096	.0100

[a]Source: McCullough (1961, p. 37).

Table 5. Size of Cochran's test, $\alpha=.01$

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0004	.0009	.0016	.0059	.0100
3	5	.0015	.0037	.0061	.0097	.0100
3	7	.0029	.0065	.0087	.0103	.0100
3	9	.0042	.0083	.0100	.0104	.0100
5	3	.0015	.0012	.0017	.0058	.0100
5	5	.0018	.0028	.0046	.0089	.0100
5	7	.0025	.0046	.0067	.0095	.0100
5	9	.0033	.0060	.0078	.0097	.0100
7	3	.0029	.0016	.0019	.0056	.0100
7	5	.0025	.0028	.0043	.0087	.0100
7	7	.0030	.0043	.0063	.0094	.0100
7	9	.0036	.0055	.0074	.0096	.0100
9	3	.0042	.0021	.0020	.0053	.0100
9	5	.0033	.0029	.0041	.0086	.0100
9	7	.0036	.0043	.0060	.0093	.0100
9	9	.0042	.0054	.0072	.0096	.0100

Table 6. Size of Welch-type test, α=.10

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0432	.0537	.0681	.0938	.1000
3	5	.0562	.0742	.0857	.0980	.1000
3	7	.0640	.0821	.0908	.0988	.1000
3	9	.0692	.0862	.0932	.0992	.1000
5	3	.0562	.0551	.0656	.0927	.1000
5	5	.0656	.0750	.0852	.0979	.1000
5	7	.0714	.0831	.0910	.0989	.1000
5	9	.0753	.0873	.0936	.0992	.1000
7	3	.0640	.0566	.0636	.0911	.1000
7	5	.0714	.0751	.0841	.0976	.1000
7	7	.0758	.0832	.0905	.0988	.1000
7	9	.0789	.0874	.0934	.0992	.1000
9	3	.0692	.0582	.0634	.0896	.1000
9	5	.0753	.0753	.0831	.0973	.1000
9	7	.0789	.0831	.0899	.0986	.1000
9	9	.0814	.0873	.0931	.0991	.1000

Table 7. Size of Cochran's test, α=.10

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0432	.0537	.0681	.0938	.1000
3	5	.0577	.0756	.0868	.0983	.1000
3	7	.0668	.0844	.0924	.0991	.1000
3	9	.0728	.0888	.0949	.0994	.1000
5	3	.0577	.0566	.0670	.0932	.1000
5	5	.0656	.0750	.0852	.0979	.1000
5	7	.0716	.0832	.0911	.0989	.1000
5	9	.0758	.0876	.0938	.0993	.1000
7	3	.0668	.0593	.0662	.0922	.1000
7	5	.0716	.0753	.0843	.0976	.1000
7	7	.0758	.0832	.0905	.0988	.1000
7	9	.0790	.0874	.0934	.0992	.1000
9	3	.0728	.0618	.0657	.0911	.1000
9	5	.0758	.0758	.0834	.0973	.1000
9	7	.0790	.0832	.0900	.0986	.1000
9	9	.0814	.0873	.0931	.0991	.1000

Table 8. Power of Cochran's test and Welch-type test for R=1, α=.05

n_1	n_2	λ	δ/σ_2	CT	WT	U^2_R
		3.5	1.53	.105	.105	0.30
3	3	9.0	2.45	.286	.286	0.62
		24.0	4.00	.695	.695	0.95
		4.2	1.50	.201	.191	0.41
3	5	15.0	2.83	.621	.607	0.90
		30.0	4.00	.877	.870	0.99
		4.8	1.51	.254	.239	0.45
3	7	15.0	2.67	.626	.609	0.96
		35.0	4.08	.906	.898	1.00
		5.1	1.51	.277	.260	0.61
3	9	15.0	2.58	.621	.601	0.97
		35.0	3.94	.894	.888	1.00
		3.0	1.11	.185		
3	13	8.0	1.81	.398		
		20.0	2.86	.708		
		3.0	1.07	.188		
3	21	8.0	1.75	.389		
		20.0	2.76	.688		
		2.5	1.15	.123	.116	0.27
5	3	7.0	1.93	.328	.317	0.60
		20.0	3.27	.739	.727	0.96
		2.6	1.02	.180	.179	0.30
5	5	8.0	1.79	.545	.544	0.70
		20.0	2.83	.937	.937	0.97
		2.8	0.98	.219	.218	0.34
5	7	7.0	1.55	.528	.526	0.66
		20.0	2.62	.954	.952	0.97
		3.0	0.97	.247	.245	0.36
5	9	8.0	1.58	.606	.601	0.74
		20.0	2.49	.955	.952	0.98
		3.0	0.91	.258		
5	13	8.0	1.49	.609		
		20.0	2.35	.950		
		3.0	0.86	.264		
5	21	8.0	1.41	.603		
		20.0	2.23	.947		

Table 8 (Continued)

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		3.0	1.20	.168	.156	0.34
7	3	7.0	1.83	.356	.337	0.64
		15.0	2.67	.629	.609	0.92
		2.0	0.83	.160	.159	0.25
7	5	5.5	1.37	.424	.421	0.56
		12.0	2.03	.786	.783	0.88
		1.8	0.72	.164	.164	0.24
7	7	6.0	1.31	.512	.512	0.61
		12.0	1.85	.840	.840	0.89
		1.8	0.68	.175	.174	0.25
7	9	6.0	1.23	.535	.533	0.62
		12.0	1.75	.859	.856	0.90
		3.0	0.81	.292		
7	13	8.0	1.33	.686		
		20.0	2.10	.982		
		3.0	0.76	.300		
7	21	8.0	1.23	.686		
		20.0	1.95	.978		
		2.0	0.94	.127	.117	0.25
9	3	7.0	1.76	.360	.341	0.66
		17.0	2.75	.666	.649	0.96
		2.0	0.79	.170	.168	0.26
9	5	6.0	1.37	.475	.471	0.60
		15.0	2.16	.880	.876	0.94
		2.0	0.71	.192	.191	0.26
9	7	5.0	1.13	.454	.453	0.55
		14.0	1.89	.912	.910	0.94
		2.0	0.67	.203	.203	0.27
9	9	5.0	1.05	.478	.478	0.56
		14.0	1.76	.928	.928	0.94
		3.0	0.75	.310		
9	13	8.0	1.23	.719		
		20.0	1.94	.989		
		3.0	0.69	.320		
9	21	8.0	1.13	.724		
		20.0	1.78	.988		

Table 8 (Continued)

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		3.0	1.11	.185		
13	3	8.0	1.81	.398		
		20.0	2.86	.708		
		3.0	0.91	.258		
13	5	8.0	1.49	.610		
		20.0	2.35	.950		
		3.0	0.81	.293		
13	7	8.0	1.33	.687		
		20.0	2.10	.983		
		3.0	0.75	.310		
13	9	8.0	1.23	.719		
		20.0	1.94	.989		
		3.0	0.68	.327		
13	13	8.0	1.11	.745		
		20.0	1.75	.992		
		3.0	0.61	.339		
13	21	8.0	1.00	.758		
		20.0	1.58	.994		
		3.0	1.07	.188		
21	3	8.0	1.75	.389		
		20.0	2.76	.687		
		3.0	0.86	.264		
21	5	8.0	1.41	.603		
		20.0	2.23	.947		
		3.0	0.76	.300		
21	7	8.0	1.23	.686		
		20.0	1.95	.978		
		3.0	0.69	.320		
21	9	8.0	1.13	.724		
		20.0	1.78	.988		
		3.0	0.61	.339		
21	13	8.0	1.00	.758		
		20.0	1.58	.994		
		3.0	0.53	.353		
21	21	8.0	0.87	.778		
		20.0	1.38	.996		

Table 9. Power of Cochran's test and Welch-type test for R=4, α=.05

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		5.0	2.89	.180	.180	0.40
3	3	14.0	4.83	.479	.479	0.80
		30.0	7.07	.793	.793	0.98
		5.2	2.82	.256	.250	0.47
3	5	16.0	4.95	.592	.586	0.91
		35.0	7.33	.858	.856	1.00
		5.5	2.85	.280	.272	0.54
3	7	20.0	5.43	.671	.665	0.97
		40.0	7.68	.888	.885	1.00
		6.0	2.94	.302	.293	0.60
3	9	23.0	5.76	.715	.711	0.99
		40.0	7.60	.884	.882	1.00
		3.0	2.06	.183		
3	13	8.0	3.36	.370		
		20.0	5.31	.661		
		3.0	2.04	.183		
3	21	8.0	3.32	.367		
		20.0	5.26	.655		
		3.5	1.99	.181	.174	0.35
5	3	9.0	3.19	.476	.462	0.71
		20.0	4.76	.845	.833	0.96
		3.6	1.90	.268	.267	0.39
5	5	10.0	3.16	.659	.660	0.80
		20.0	4.47	.927	.927	0.97
		3.8	1.89	.304	.303	0.43
5	7	7.0	2.57	.519	.519	0.66
		20.0	4.34	.929	.929	0.98
		4.0	1.91	.327	.325	0.45
5	9	9.0	2.86	.634	.632	0.78
		20.0	4.27	.928	.928	0.98
		3.0	1.62	.260		
5	13	8.0	2.65	.583		
		20.0	4.19	.925		
		3.0	1.59	.262		
5	21	8.0	2.60	.581		
		20.0	4.12	.922		

Table 9 (Continued)

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		3.0	1.65	.175	.161	0.34
7	3	7.0	2.52	.409	.384	0.64
		15.0	3.68	.747	.723	0.92
		2.5	1.39	.211	.210	0.30
7	5	6.5	2.24	.524	.522	0.63
		13.0	3.17	.848	.847	0.90
		2.5	1.34	.234	.234	0.31
7	7	7.0	2.24	.591	.591	0.67
		12.0	2.93	.837	.837	0.89
		2.5	1.31	.244	.244	0.32
7	9	7.5	2.26	.633	.633	0.71
		12.0	2.86	.841	.841	0.90
		3.0	1.39	.296		
7	13	8.0	2.28	.668		
		20.0	3.60	.971		
		3.0	1.36	.300		
7	21	8.0	2.23	.667		
		20.0	3.52	.969		
		3.0	1.53	.183	.166	0.36
9	3	7.0	2.33	.417	.388	0.66
		15.0	3.42	.742	.713	0.94
		3.0	1.39	.263	.261	0.36
9	5	6.0	1.97	.510	.507	0.60
		15.0	3.11	.917	.915	0.94
		2.5	1.21	.247	.247	0.32
9	7	6.0	1.88	.551	.550	0.62
		14.0	2.87	.918	.918	0.94
		2.5	1.18	.258	.258	0.29
9	9	6.0	1.83	.565	.565	0.63
		14.0	2.79	.922	.922	0.94
		3.0	1.25	.315		
9	13	8.0	2.04	.708		
		20.0	3.23	.984		
		3.0	1.21	.320		
9	21	8.0	1.98	.709		
		20.0	3.14	.983		

Table 9 (Continued)

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		3.0	1.39	.191		
13	3	8.0	2.26	.465		
		20.0	3.58	.820		
		3.0	1.23	.272		
13	5	8.0	2.02	.663		
		20.0	3.19	.977		
		3.0	1.16	.306		
13	7	8.0	1.90	.715		
		20.0	3.00	.989		
		3.0	1.12	.321		
13	9	8.0	1.83	.733		
		20.0	2.89	.991		
		3.0	1.07	.334		
13	13	8.0	1.75	.745		
		20.0	2.77	.992		
		3.0	1.03	.341		
13	21	8.0	1.69	.749		
		20.0	2.67	.991		
		3.0	1.25	.321		
21	3	8.0	2.05	.733		
		20.0	3.24	.991		
		3.0	1.08	.334		
21	5	8.0	1.77	.745		
		20.0	2.79	.992		
		3.0	1.00	.341		
21	7	8.0	1.63	.749		
		20.0	2.58	.991		
		3.0	0.95	.194		
21	9	8.0	1.55	.445		
		20.0	2.46	.778		
		3.0	0.90	.275		
21	13	8.0	1.46	.658		
		20.0	2.31	.972		
		3.0	0.85	.312		
21	21	8.0	1.38	.724		
		20.0	2.18	.990		

Table 10. Power of Cochran's test and Welch-type test for R=10, α=.05

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		6.0	4.69	.249	.249	0.47
3	3	16.0	7.66	.553	.553	0.84
		35.0	11.33	.837	.837	0.99
		6.2	4.68	.298	.295	0.55
3	5	17.0	7.75	.598	.596	0.93
		40.0	11.89	.877	.877	1.00
		6.4	4.72	.310	.306	0.60
3	7	20.0	8.34	.655	.652	0.97
		40.0	11.79	.875	.874	1.00
		6.5	4.73	.314	.310	0.63
3	9	23.0	8.90	.701	.699	0.99
		40.0	11.74	.873	.872	1.00
		3.0	3.20	.182		
3	13	8.0	5.22	.363		
		20.0	8.26	.650		
		3.0	3.18	.182		
3	21	8.0	5.20	.361		
		20.0	8.22	.648		
		4.5	3.24	.278	.270	0.43
5	3	9.0	4.58	.539	.528	0.71
		20.0	6.83	.891	.885	0.96
		4.6	3.18	.354	.354	0.47
5	5	9.0	4.45	.616	.616	0.75
		20.0	6.63	.920	.921	0.97
		4.8	3.21	.380	.380	0.51
5	7	7.0	3.87	.519	.518	0.66
		20.0	6.55	.921	.921	0.98
		4.0	2.91	.329	.329	0.45
5	9	10.0	4.59	.671	.671	0.82
		20.0	6.50	.920	.920	0.98
		3.0	2.50	.261		
5	13	8.0	4.08	.577		
		20.0	6.45	.919		
		3.0	2.48	.262		
5	21	8.0	4.05	.576		
		20.0	6.40	.917		

Table 10 (Continued)

n_1	n_2	λ	δ/σ_2	CT	WT	U^2_R
		3.0	2.30	.205	.191	0.34
7	3	7.0	3.51	.474	.452	0.64
		15.0	5.14	.832	.814	0.92
		3.0	2.21	.272	.271	0.36
7	5	7.5	3.49	.611	.611	0.69
		14.0	4.77	.881	.881	0.92
		3.0	2.17	.288	.288	0.36
7	7	7.5	3.43	.627	.627	0.70
		12.0	4.34	.833	.833	0.89
		3.0	2.15	.294	.294	0.37
7	9	8.0	3.51	.661	.661	0.75
		12.0	4.30	.835	.835	0.90
		3.0	2.13	.298		
7	13	8.0	3.47	.663		
		20.0	5.49	.967		
		3.0	2.10	.300		
7	21	8.0	3.44	.662		
		20.0	5.43	.966		
		3.0	2.08	.211	.194	0.36
9	3	7.0	3.18	.487	.458	0.66
		15.0	4.65	.839	.814	0.94
		3.0	1.98	.286	.284	0.36
9	5	6.5	2.92	.576	.574	0.64
		15.0	4.43	.928	.927	0.94
		2.5	1.77	.260	.260	0.32
9	7	7.0	2.96	.633	.633	0.68
		14.0	4.19	.917	.917	0.94
		2.5	1.75	.267	.267	0.29
9	9	7.0	2.92	.640	.640	0.69
		14.0	4.14	.919	.919	0.94
		3.0	1.89	.319		
9	13	8.0	3.08	.705		
		20.0	4.87	.982		
		3.0	1.86	.321		
9	21	8.0	3.04	.705		
		20.0	4.81	.981		

Table 10 (Continued)

n_1	n_2	λ	δ/σ_2	CT	WT	U_R^2
		3.0	1.82	.213		
13	3	8.0	2.97	.544		
		20.0	4.70	.913		
		3.0	1.71	.296		
13	5	8.0	2.78	.701		
		20.0	4.40	.986		
		3.0	1.65	.322		
13	7	8.0	2.70	.730		
		20.0	4.27	.990		
		3.0	1.63	.331		
13	9	8.0	2.65	.740		
		20.0	4.20	.991		
		3.0	1.59	.339		
13	13	8.0	2.60	.745		
		20.0	4.11	.991		
		3.0	1.57	.343		
13	21	8.0	2.56	.747		
		20.0	4.04	.990		
		3.0	1.56	.210		
21	3	8.0	2.54	.517		
		20.0	4.02	.874		
		3.0	1.42	.298		
21	5	8.0	2.33	.706		
		20.0	3.68	.987		
		3.0	1.36	.330		
21	7	8.0	2.23	.749		
		20.0	3.52	.993		
		3.0	1.33	.343		
21	9	8.0	2.17	.763		
		20.0	3.43	.994		
		3.0	1.29	.354		
21	13	8.0	2.10	.773		
		20.0	3.33	.995		
		3.0	1.25	.360		
21	21	8.0	2.05	.777		
		20.0	3.24	.995		

Table 11. Size of unilateral Welch-type test, α=.05[a]

n_1	n_2	r_2'	R=1	R=4	R=10	R=100	R=∞
3	3	0.388	.0500	.0450	.0465	.0494	.0500
3	5	0.082	.0500	.0479	.0487	.0498	.0500
3	7	0.034	.0500	.0488	.0493	.0499	.0500
3	9	0.019	.0500	.0491	.0495	.0499	.0500
5	3	0.911	.0500	.0389	.0407	.0482	.0500
5	5	0.167	.0500	.0453	.0469	.0495	.0500
5	7	0.068	.0500	.0474	.0484	.0498	.0500
5	9	0.037	.0500	.0483	.0490	.0499	.0500
7	3	1.278	.0500	.0357	.0364	.0466	.0500
7	5	0.215	.0500	.0433	.0451	.0492	.0500
7	7	0.086	.0500	.0461	.0475	.0496	.0500
7	9	0.046	.0500	.0475	.0485	.0498	.0500
9	3	1.537	.0500	.0342	.0335	.0451	.0500
9	5	0.246	.0500	.0412	.0435	.0489	.0500
9	7	0.096	.0500	.0454	.0467	.0495	.0500
9	9	0.052	.0500	.0468	.0479	.0497	.0500

[a]Source: McCullough (1961, p. 36).

Table 12. Size of unilateral Cochran's test, α=.05

n_1	n_2	t_2'	R=1	R=4	R=10	R=100	R=∞
3	3	1.8178	.0500	.0470	.0491	.0506	.0500
3	5	1.6622	.0500	.0512	.0520	.0508	.0500
3	7	1.6411	.0500	.0522	.0523	.0507	.0500
3	9	1.6454	.0500	.0523	.0522	.0506	.0500
5	3	2.3525	.0500	.0389	.0408	.0482	.0500
5	5	1.8955	.0500	.0459	.0475	.0497	.0500
5	7	1.7865	.0500	.0482	.0491	.0499	.0500
5	9	1.7412	.0500	.0492	.0497	.0500	.0500
7	3	2.7788	.0500	.0357	.0364	.0466	.0500
7	5	2.0865	.0500	.0434	.0452	.0492	.0500
7	7	1.9270	.0500	.0464	.0477	.0497	.0500
7	9	1.8580	.0500	.0478	.0487	.0498	.0500
9	3	3.0825	.0500	.0345	.0338	.0452	.0500
9	5	2.2165	.0500	.0419	.0435	.0489	.0500
9	7	2.0240	.0500	.0454	.0467	.0495	.0500
9	9	1.9403	.0500	.0470	.0481	.0497	.0500

Table 13. Constants for optimal conservative unilateral preliminary testing procedure, Y_1', $\alpha_2=\alpha=.05$[a]

n_1	n_2	α_0^*	a_0	t^2	k
3	3	.39	1.564	5.787	0.313
3	5	.58	0.313	3.336	0.096
3	7	.64	0.160	2.704	0.052
5	3	.58	1.842	18.382	4.239
5	5	.13	3.427	5.099	0.662
7	3	.16	16.711	7.054	2.945

[a]Source: McCullough (1961, p. 92).

Table 14. Constants for optimal conservative unilateral preliminary testing procedure, Y_2', $\alpha_2=\alpha=.05$[a]

n_1	n_2	α_0^*	a_0	t^2	r_1	r_2
3	3	.51	0.961	5.397	3.086	0.388
3	5	.73	0.170	2.772	3.086	0.082
3	7	.83	0.064	1.806	3.086	0.034
5	3	.80	0.809	8.434	0.385	0.911
5	5	.19	2.585	5.059	0.385	0.167
7	3	.80	1.408	12.745	0.143	1.278

[a]Source: Gurland and McCullough (1962, p. 411).

Table 15. Constants for optimal conservative unilateral preliminary testing procedure, Y_3', $\alpha_2=\alpha=.05$

n_1	n_2	α_0^*	a_0	t^2	t_1	t_2
3	3	.39	1.564	5.785	4.3027	4.3027
3	5	.66	0.231	3.045	4.3027	2.7764
3	7	.77	0.091	2.078	4.3027	2.4469
5	3	.10	18.735	6.123	2.7764	4.3027
5	5	.13	3.426	5.099	2.7764	2.7764
7	3	.20	13.038	6.026	2.4469	4.3027

Table 16. Size of optimal conservative unilateral preliminary testing procedure, Y_1', $\alpha_2=\alpha=.05$[a]

n_1	n_2	R=1	R=2	R=4	R=10	R=40	R=100	R=∞
3	3	.0500	.0484	.0492	.0495	.0481	.0470	.0500
3	5	.0500	.0449	.0418	.0384	.0422	.0464	.0500
3	7	.0500	.0480	.0415	.0297	.0280	.0358	.0500
5	3	.0500	.0453	.0438	.0460	.0491	.0498	.0500
5	5	.0500	.0493	.0497	.0486	.0473	.0483	.0500
7	3	.0500	.0406	.0367	.0400	.0489	.0500	.0500

[a]Source: McCullough (1961, p. 93).

Table 17. Size of optimal conservative unilateral preliminary testing procedure, Y_2', $\alpha_2=\alpha=.05$[a]

n_1	n_2	R=1	R=4	R=10	R=100	R=∞
3	3	.0500	.0485	.0498	.0498	.0500
3	5	.0500	.0499	.0498	.0499	.0500
3	7	.0500	.0497	.0497	.0499	.0500
5	3	.0500	.0388	.0407	.0482	.0500
5	5	.0500	.0496	.0497	.0496	.0500
7	3	.0500	.0356	.0363	.0467	.0500

[a]Source: Gurland and McCullough (1962, p. 413).

Table 18. Size of optimal conservative unilateral preliminary testing procedure, Y_3', $\alpha_2=\alpha=.05$

n_1	n_2	R=1	R=2	R=4	R=10	R=40	R=100	R=∞
3	3	.0500	.0483	.0492	.0495	.0469	.0470	.0500
3	5	.0500	.0498	.0490	.0477	.0482	.0490	.0500
3	7	.0500	.0494	.0486	.0483	.0492	.0496	.0500
5	3	.0500	.0384	.0336	.0334	.0392	.0438	.0500
5	5	.0500	.0492	.0497	.0486	.0473	.0483	.0500
7	3	.0500	.0338	.0270	.0267	.0355	.0424	.0500

Table 19. Constants for optimal conservative bilateral preliminary testing procedure, Y_2, $\alpha_1=\alpha_2=\alpha=.05$

n_1	n_2	α_0^*	a_1	a_2	t	t_1	t_2
3	3	.47	0.307	3.255	2.7764	4.3027	4.3027
3	5	.75	0.265	0.633	2.4469	4.3027	2.7764
3	7	.88	0.213	0.315	2.3060	4.3027	2.4469
3	13	.98	0.119	0.126	2.1448	4.3027	2.1788
5	3	.75	1.580	3.775	2.4469	2.7764	4.3027
5	5	.19	0.235	4.251	2.3060	2.7764	2.7764
5	9	.55	0.254	0.778	2.1788	2.7764	2.3060
7	3	.88	3.177	4.690	2.3060	2.4469	4.3027
7	7	.09	0.222	4.495	2.1788	2.4469	2.4469
7	13	.47	0.273	0.789	2.1009	2.4469	2.1788
9	5	.55	1.285	3.943	2.1788	2.3060	2.7764
9	9	.06	0.241	4.154	2.1199	2.3060	2.3060
13	3	.98	7.942	8.418	2.1448	2.1788	4.3027
13	7	.47	1.267	3.669	2.1009	2.1788	2.4469
13	13	.03	0.267	3.752	2.0639	2.1788	2.1788

Table 20. Size of optimal conservative bilateral preliminary testing procedure, Y_2, $\alpha_1=\alpha_2=\alpha=.05$

n_1	n_2	R=1	R=2	R=4	R=10	R=40	R=100	R=∞
3	3	.0324	.0343	.0393	.0468	.0497	.0488	.0500
3	5	.0298	.0372	.0448	.0497	.0496	.0495	.0500
3	7	.0315	.0396	.0463	.0496	.0498	.0498	.0500
3	13	.0375	.0436	.0475	.0495	.0500	.0500	.0500
5	3	.0298	.0254	.0246	.0278	.0375	.0434	.0500
5	5	.0451	.0464	.0489	.0496	.0478	.0484	.0500
5	9	.0421	.0481	.0498	.0483	.0488	.0494	.0500
7	3	.0315	.0253	.0228	.0250	.0356	.0423	.0500
7	7	.0483	.0491	.0500	.0485	.0479	.0489	.0500
7	13	.0456	.0499	.0489	.0477	.0491	.0496	.0500
9	5	.0421	.0361	.0338	.0367	.0445	.0475	.0500
9	9	.0491	.0495	.0494	.0474	.0482	.0492	.0500
13	3	.0375	.0309	.0262	.0248	.0324	.0397	.0500
13	7	.0456	.0394	.0374	.0406	.0465	.0485	.0500
13	13	.0497	.0498	.0488	.0471	.0488	.0495	.0500

Table 21. Power of optimal conservative bilateral preliminary testing procedure, γ_2, $\alpha_1=\alpha_2=\alpha=.05$

		R=1			R=4			R=10		
n_1	n_2	λ	δ/σ_2	γ_2	λ	δ/σ_2	γ_2	λ	δ/σ_2	γ_2
		3.5	1.53	.207	5.0	2.89	.287	6.0	4.69	.311
3	3	9.0	2.45	.463	14.0	4.83	.588	16.0	7.66	.580
		24.0	4.00	.831	30.0	7.07	.829	35.0	11.33	.840
		4.2	1.50	.258	5.2	2.82	.292	6.2	4.68	.307
3	5	15.0	2.83	.688	16.0	4.95	.602	17.0	7.75	.599
		30.0	4.00	.894	35.0	7.33	.858	40.0	11.89	.877
		4.8	1.51	.289	5.5	2.85	.295	6.4	4.72	.312
3	7	15.0	2.67	.659	20.0	5.43	.671	20.0	8.34	.655
		35.0	4.08	.907	40.0	7.68	.888	40.0	11.79	.875
		3.0	1.11	.190	3.0	2.06	.186	3.0	3.20	.182
3	13	8.0	1.81	.406	8.0	3.36	.371	8.0	5.22	.363
		20.0	2.86	.710	20.0	5.31	.661	20.0	8.26	.650
		2.5	1.15	.163	3.5	1.99	.213	4.5	3.24	.293
5	3	7.0	1.93	.405	9.0	3.19	.525	9.0	4.58	.554
		20.0	3.27	.786	20.0	4.76	.873	20.0	6.83	.895
		2.6	1.02	.265	3.6	1.90	.335	4.6	3.18	.380
5	5	8.0	1.79	.672	10.0	3.16	.718	9.0	4.45	.632
		20.0	2.83	.971	20.0	4.47	.940	20.0	6.63	.922
		3.0	0.97	.296	4.0	1.91	.346	4.0	2.91	.332
5	9	8.0	1.58	.666	9.0	2.86	.642	10.0	4.59	.672
		20.0	2.49	.964	20.0	4.27	.928	20.0	6.50	.920
		3.0	1.20	.193	3.0	1.65	.187	3.0	2.30	.209
7	3	7.0	1.83	.396	7.0	2.52	.431	7.0	3.51	.481
		15.0	2.67	.659	15.0	3.68	.772	15.0	5.14	.837
		1.8	0.72	.219	2.5	1.34	.277	3.0	2.17	.302
7	7	6.0	1.31	.603	7.0	2.24	.639	7.5	3.43	.637
		12.0	1.85	.896	12.0	2.93	.862	12.0	4.34	.837
		3.0	0.81	.331	3.0	1.39	.307	3.0	2.13	.299
7	13	8.0	1.33	.728	8.0	2.28	.674	8.0	3.47	.663
		20.0	2.10	.986	20.0	3.60	.973	20.0	5.49	.967
		2.0	0.79	.209	3.0	1.39	.277	3.0	1.98	.288
9	5	6.0	1.37	.538	6.0	1.97	.528	6.5	2.92	.578
		15.0	2.16	.906	15.0	3.11	.924	15.0	4.43	.928
		2.0	0.67	.249	2.5	1.18	.288	2.5	1.75	.274
9	9	5.0	1.05	.545	6.0	1.83	.599	7.0	2.92	.644
		14.0	1.76	.951	14.0	2.79	.932	14.0	4.14	.920
		3.0	1.11	.190	3.0	1.39	.192	3.0	1.82	.213
13	3	8.0	1.81	.406	8.0	2.26	.469	8.0	2.97	.545
		20.0	2.86	.710	20.0	3.58	.825	20.0	4.70	.914
		3.0	0.81	.331	3.0	1.16	.311	3.0	1.65	.322
13	7	8.0	1.33	.728	8.0	1.90	.722	8.0	2.70	.731
		20.0	2.10	.986	20.0	3.00	.991	20.0	4.27	.990
		3.0	0.68	.367	3.0	1.07	.352	3.0	1.59	.341
13	13	8.0	1.11	.783	8.0	1.75	.760	8.0	2.60	.748
		20.0	1.75	.995	20.0	2.77	.993	20.0	4.11	.993

Table 22. Comparison of Monte Carlo and exact computations

f_1	f_2	θ	$\overline{P}_E(\Lambda_E,\theta)$	$P_E(\Lambda_E,\theta)$
6	6	0.00	.0500	.0500
		3.00	.255	.253
		6.75	.520	.514
		12.00	.776	.772
6	12	0.00	.0514	.0500
		3.00	.273	.275
		6.75	.559	.557
		12.00	.817	.816
6	18	0.00	.0513	.0500
		3.00	.285	.286
		6.75	.580	.578
		12.00	.836	.835
12	12	0.00	.0513	.0500
		3.00	.285	.286
		6.75	.580	.578
		12.00	.836	.835

Table 23. Size of bivariate Cochran's test for $(f_1,f_2) = (6,6)$, $\alpha=.05$

λ_1	λ_2	MCT	APDF	95% conf. int. MCT		95% conf. int. APDF	
.01	.01	.0478	.0551	.0403	.0553	.0463	.0639
.10	.10	.0298	.0579	.0262	.0335	.0514	.0644
.25	.25	.0188	.0489	.0172	.0203	.0457	.0521
.33	.33	.0162	.0456	.0152	.0172	.0436	.0476
.50	.50	.0145	.0435	.0141	.0150	.0427	.0442
.67	.67	.0164	.0465	.0154	.0173	.0444	.0486
.90	.90	.0308	.0598	.0269	.0346	.0530	.0666
.96	.96	.0419	.0606	.0358	.0479	.0520	.0692
.99	.99	.0486	.0560	.0412	.0560	.0474	.0646
.10	.90	.0207	.0490	.0173	.0241	.0435	.0545
.25	.75	.0168	.0466	.0148	.0188	.0433	.0498
.50	.90	.0201	.0494	.0173	.0229	.0448	.0540
.75	.25	.0153	.0441	.0138	.0167	.0414	.0468
.90	.10	.0176	.0439	.0152	.0199	.0395	.0482
.90	.50	.0186	.0476	.0162	.0210	.0437	.0515

Table 24. Size of bivariate Cochran's test for $(f_1,f_2) = (6,12)$, $\alpha=.05$

λ_1	λ_2	MCT	APDF	95% conf. int. MCT		95% conf. int. APDF	
.01	.01	.0500	.0517	.0467	.0533	.0482	.0553
.10	.10	.0420	.0530	.0401	.0439	.0504	.0557
.25	.25	.0323	.0504	.0318	.0329	.0495	.0513
.33	.33	.0284	.0495	.0278	.0291	.0492	.0499
.50	.50	.0234	.0505	.0220	.0249	.0487	.0523
.67	.67	.0221	.0554	.0198	.0244	.0516	.0593
.90	.90	.0320	.0655	.0273	.0367	.0572	.0739
.96	.96	.0424	.0625	.0357	.0491	.0530	.0720
.99	.99	.0491	.0565	.0410	.0571	.0473	.0658
.10	.90	.0300	.0513	.0258	.0341	.0462	.0563
.25	.75	.0260	.0520	.0235	.0286	.0486	.0554
.50	.90	.0236	.0565	.0199	.0272	.0508	.0621
.75	.25	.0247	.0505	.0224	.0270	.0475	.0535
.90	.10	.0282	.0493	.0246	.0318	.0451	.0535
.90	.50	.0220	.0554	.0188	.0252	.0504	.0604

Table 25. Size of bivariate Cochran's test for $(f_1,f_2) = (6,18)$, $\alpha=.05$

λ_1	λ_2	MCT	APDF	95% conf. int. MCT		95% conf. int. APDF	
.01	.01	.0521	.0531	.0496	.0546	.0505	.0557
.10	.10	.0460	.0529	.0448	.0472	.0513	.0544
.25	.25	.0370	.0507	.0364	.0376	.0505	.0510
.33	.33	.0329	.0504	.0318	.0340	.0495	.0513
.50	.50	.0263	.0522	.0243	.0284	.0497	.0547
.67	.67	.0233	.0576	.0205	.0262	.0529	.0622
.90	.90	.0415	.0620	.0346	.0485	.0521	.0719
.96	.96	.0313	.0662	.0263	.0363	.0572	.0751
.99	.99	.0481	.0556	.0398	.0564	.0461	.0651
.10	.90	.0336	.0513	.0290	.0382	.0464	.0563
.25	.75	.0292	.0530	.0262	.0322	.0492	.0568
.50	.90	.0238	.0578	.0199	.0278	.0517	.0639
.75	.25	.0280	.0520	.0252	.0308	.0487	.0554
.90	.10	.0321	.0504	.0279	.0363	.0461	.0546
.90	.50	.0222	.0567	.0186	.0258	.0513	.0622

Table 26. Size of bivariate Cochran's test for $(f_1,f_2) = (12,12)$, $\alpha=.05$

λ_1	λ_2	MCT	APDF	95% conf. int. MCT		95% conf. int. APDF	
.01	.01	.0488	.0506	.0449	.0528	.0464	.0547
.10	.10	.0415	.0533	.0389	.0441	.0498	.0568
.25	.25	.0344	.0526	.0331	.0358	.0506	.0547
.33	.33	.0324	.0519	.0315	.0332	.0505	.0532
.50	.50	.0306	.0510	.0304	.0309	.0508	.0512
.67	.67	.0321	.0516	.0313	.0329	.0502	.0529
.90	.90	.0465	.0521	.0428	.0501	.0479	.0564
.96	.96	.0414	.0532	.0387	.0441	.0496	.0569
.99	.99	.0491	.0509	.0449	.0533	.0465	.0553
.10	.90	.0342	.0501	.0317	.0366	.0469	.0534
.25	.75	.0318	.0510	.0305	.0332	.0490	.0530
.50	.90	.0339	.0508	.0321	.0357	.0484	.0533
.75	.25	.0314	.0503	.0302	.0325	.0486	.0520
.90	.10	.0330	.0487	.0309	.0351	.0459	.0515
.90	.50	.0336	.0514	.0319	.0353	.0491	.0537

Table 27. Power of bivariate Cochran's test for $(f_1,f_2) = (6,6)$, α=.05

λ_1	λ_2	θ	MCT	APDF	λ_1	λ_2	θ	MCT	APDF
		3.00	.182	.194			3.00	.128	.231
.01	.01	6.75	.376	.391	.10	.90	6.75	.316	.487
		12.00	.603	.616			12.00	.571	.754
		3.00	.152	.256			3.00	.119	.237
.10	.10	6.75	.346	.445	.25	.75	6.75	.307	.497
		12.00	.585	.676			12.00	.563	.757
		3.00	.125	.233			3.00	.130	.239
.25	.25	6.75	.314	.478	.50	.90	6.75	.324	.493
		12.00	.564	.724			12.00	.577	.746
		3.00	.117	.233			3.00	.114	.232
.33	.33	6.75	.304	.486	.75	.25	6.75	.301	.491
		12.00	.557	.740			12.00	.556	.753
		3.00	.112	.234			3.00	.121	.224
.50	.50	6.75	.299	.496	.90	.10	6.75	.307	.478
		12.00	.555	.755			12.00	.560	.744
		3.00	.120	.240			3.00	.127	.235
.67	.67	6.75	.313	.498	.90	.50	6.75	.319	.489
		12.00	.568	.752			12.00	.574	.744
		3.00	.163	.241					
.90	.90	6.75	.366	.470					
		12.00	.609	.702					
		3.00	.184	.228					
.96	.96	6.75	.388	.441					
		12.00	.623	.669					
		3.00	.197	.211					
.99	.99	6.75	.400	.416					
		12.00	.630	.644					

Table 28. Power of bivariate Cochran's test for $(f_1,f_2) = (6,12)$, $\alpha=.05$

λ_1	λ_2	θ	MCT	APDF	λ_1	λ_2	θ	MCT	APDF
		3.00	.240	.244			3.00	.171	.252
.01	.01	6.75	.506	.510	.10	.90	6.75	.400	.526
		12.00	.767	.771			12.00	.671	.793
		3.00	.225	.257			3.00	.165	.261
.10	.10	6.75	.494	.532	.25	.75	6.75	.395	.539
		12.00	.763	.791			12.00	.669	.798
		3.00	.200	.261			3.00	.139	.257
.25	.25	6.75	.463	.546	.50	.90	6.75	.341	.516
		12.00	.742	.807			12.00	.599	.764
		3.00	.185	.262			3.00	.163	.259
.33	.33	6.75	.440	.548	.75	.25	6.75	.395	.540
		12.00	.721	.809			12.00	.669	.799
		3.00	.160	.263			3.00	.171	.251
.50	.50	6.75	.392	.544	.90	.10	6.75	.400	.529
		12.00	.667	.801			12.00	.671	.794
		3.00	.145	.264			3.00	.139	.256
.67	.67	6.75	.357	.531	.90	.50	6.75	.341	.517
		12.00	.620	.777			12.00	.600	.765
		3.00	.163	.248					
.90	.90	6.75	.366	.475					
		12.00	.608	.703					
		3.00	.182	.228					
.96	.96	6.75	.384	.440					
		12.00	.619	.666					
		3.00	.194	.208					
.99	.99	6.75	.396	.412					
		12.00	.626	.640					

Table 29. Power of bivariate Cochran's test for $(f_1,f_2) = (6,18)$, $\alpha=.05$

λ_1	λ_2	θ	MCT	APDF	λ_1	λ_2	θ	MCT	APDF
		3.00	.267	.269			3.00	.190	.263
.01	.01	6.75	.554	.557	.10	.90	6.75	.431	.543
		12.00	.815	.816			12.00	.703	.807
		3.00	.232	.278			3.00	.184	.274
.25	.25	6.75	.516	.575	.25	.75	6.75	.428	.558
		12.00	.790	.832			12.00	.702	.813
		3.00	.179	.277			3.00	.182	.271
.50	.50	6.75	.425	.564	.75	.25	6.75	.426	.557
		12.00	.700	.816			12.00	.700	.813
		3.00	.163	.251			3.00	.189	.260
.90	.90	6.75	.367	.479	.90	.10	6.75	.428	.544
		12.00	.611	.708			12.00	.699	.807
		3.00	.194	.208					
.99	.99	6.75	.396	.412					
		12.00	.628	.642					

Table 30. Power of bivariate Cochran's test for $(f_1,f_2) = (12,12)$, $\alpha=.05$

λ_1	λ_2	θ	MCT	APDF	λ_1	λ_2	θ	MCT	APDF
		3.00	.239	.244			3.00	.215	.271
.01	.01	6.75	.506	.511	.10	.90	6.75	.488	.566
		12.00	.769	.772			12.00	.769	.830
		3.00	.213	.271			3.00	.210	.277
.25	.25	6.75	.486	.560	.25	.75	6.75	.484	.573
		12.00	.764	.818			12.00	.766	.832
		3.00	.206	.278			3.00	.207	.274
.50	.50	6.75	.481	.575	.75	.25	6.75	.482	.570
		12.00	.764	.832			12.00	.764	.830
		3.00	.236	.270			3.00	.211	.267
.90	.90	6.75	.509	.549	.90	.10	6.75	.484	.562
		12.00	.774	.802			12.00	.765	.826
		3.00	.251	.256					
.99	.99	6.75	.521	.526					
		12.00	.777	.781					